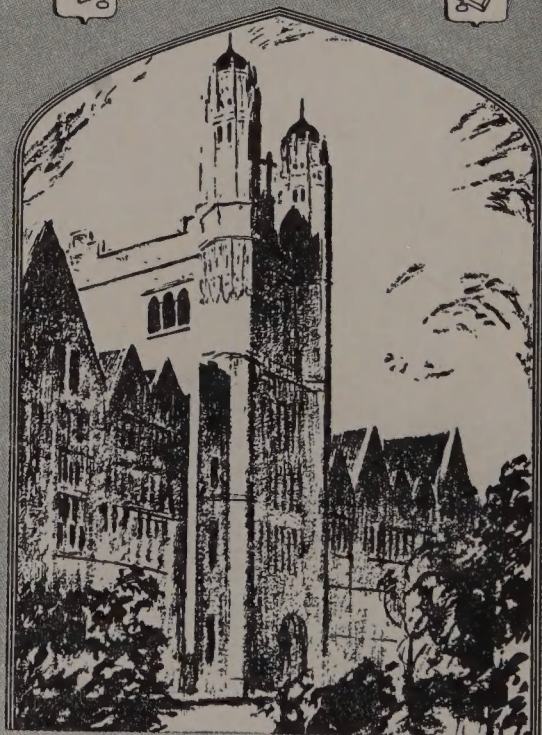# MODERN PERSPECTIVE:

A TREATISE

UPON THE

PRINCIPLES AND PRACTICE OF PLANE AND CYLINDRICAL PERSPECTIVE.

BY

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PREFACE.

IN compliance with the wishes of my pupils, and in fulfilment of a promise made some time since to the publishers of the *American Architect and Building News*, I have collected and revised a series of papers upon Perspective, which I five years ago contributed to the columns of that journal, adding half-a-dozen chapters and a dozen pages of illustrations.

A new treatment of so old a theme would be uncalled for, did not even the more elaborate treatises seem to be deficient in comprehensiveness and scientific simplicity, while the practical handbooks fail to make the reader acquainted with some methods that are found in experience to be among the most convenient and practical of all. Most of what I have to say is, of course, in substance, an old story; but it is a story which can, I think, be told again with profit, so as the better to lead up to the matter that is comparatively new. That I have anything to offer which is absolutely new, that I have in my explorations found any field absolutely untrodden by my predecessors, I can hardly suppose : I am too used, in these regions, to discover the footprints

of unknown or forgotten pioneers in what I had taken
to be really *terra incognita.* But I am sure that if the
reader will accompany me, he will come to some things
that, if not absolutely novel, are new to him, and that
he will reach some points of view from which the more
familiar ground will present an unaccustomed aspect.

This discussion of the subject differs from that gener-
ally given, in several particulars; much greater promi-
nence being assigned to the phenomena of parallel
planes than is usual, and use being made of the laws
thus established to determine the perspective of lines
of intersection and of shadows, — subjects that seem
hitherto to have received but little attention.

The perspective of divergent lines, also, and of shad-
ows cast by divergent rays, as from an artificial source
of light, is a subject that seems to have been almost
entirely neglected.

In the course of these investigations it will appear
that the horizontal plane hardly deserves the paramount
importance commonly assigned to it, and that the prac-
tice of referring all constructions to that plane is pro-
ductive of needless inconvenience. The well-known
method, also, of points of distance, and points of meas-
ures, which is generally treated as an auxiliary method
of but limited serviceability, will be shown to be of
universal application, and to suffice for the solution of
almost all problems. The development of this method

to its legitimate results leads to the consistent use of the Perspective Plan, rendering unnecessary the construction of the orthographic plan, by the aid of which perspective drawings are commonly made.

Any treatise on perspective is, of course, mainly directed to meet the wants of the architect; and the problems with which he deals are free from most of the perplexities that constantly annoy the student of nature. But there are difficulties and apparent anomalies which confuse the mind even of the architectural draughtsman, and in disposing of these it is possible also to explain the discrepancies which are found to exist between sketches made faithfully from nature and drawings made according to the common perspective rules, — discrepancies which have naturally produced among artists a certain disregard and contempt for the rules themselves. It will be shown, as indeed hardly needs to be pointed out, that in drawing from nature one works, virtually, not upon a plane, but upon a cylinder. The discussion of Plane Perspective needs to be supplemented, then, by chapters on Cylindrical, or, as it is sometimes called, Panoramic Perspective; and an explanation of the principles and rules of this method show its results to be exactly conformable, in kind, to those reached when drawing merely by the eye. Much that I have to say is accordingly as pertinent to the work of the landscape painter or the historical painter

as to that of the architect. Indeed, the questions that
arise when the human figure is to be drawn in per-
spective mainly concern them.

A separate chapter discusses certain methods em-
ployed for limiting the space required for making draw-
ings in perspective, especially that of the late M. Adhé-
mar, — methods of the greatest value when, as in fresco-
painting or scene-painting, the picture is large compared
with the size of the room in which it is to be made. I
have taken the liberty of considerably modifying the
details of M. Adhémar's processes, and in an impor-
tant particular have suggested an alternative procedure,
which, if not intrinsically preferable to his own, is at
least more in accord with the point of view taken in
this work.

To this chapter I have added a chapter upon the in-
terpretation of Perspective Drawings. Photography has
given to the discussion of this subject an importance
which it did not previously possess, for it is often de-
sirable to obtain from the perspective view taken by
the camera the real proportions or dimensions of the
object shown. This is sometimes impossible, sufficient
data not being furnished by the picture itself, and no
other information being accessible. But when it is pos-
sible it is not difficult, as I have endeavored to make
plain.

In all these chapters I have avoided a too formal
method of demonstration, using a somewhat conversa-

tional style, and endeavoring to make the subject intelligible without employing the apparatus of theorems and problems. For those, however, whose tastes or habits of mind might demand a more concise and formal treatment, I have added a couple of chapters, in the first of which are collected the geometrical principles involved in the preceding pages, while the second contains, in the form of geometrical problems, solutions not only of the simple questions which the ordinary practice of perspective drawing presents, but of most of the more elementary problems of Descriptive Geometry.

Finally, to meet the practical needs of the practical draughtsman, I have added, at the end of the book, a chapter upon The Practical Problem, showing just how one goes to work to lay out the main lines of a perspective drawing according to the system presented in the previous chapters. To these the student is referred for the further illustration of matters of detail.

These last three chapters are, in a sense, quite independent of what has preceded them, and might about as well have come first as last. Some readers may find an advantage in first obtaining a general view of the whole subject by their aid, before undertaking to follow the more detailed treatment presented in the body of the work.

The plates that accompany the text have taken, altogether, quite as much time and labor in their prep-

aration as has the text itself, — a labor which has been
lightened to me by the intelligent co-operation of a
score of young men, most of them my pupils, who
have, one after another, taken their turn at what has
seemed a never-ending task.   Among these I owe my
acknowledgments for something more than merely cleri-
cal service to Messrs. H. F. Burr, A. H. Munsell, A. B.
Harlow, G. L. Heins, F. D. Sherman, and G. T. Snelling.
Especially ought I to mention the name of Mr. A. J.
Boyden, to whose intelligence and skill nearly half the
series bear witness.

I cannot lay down my pen without acknowledging
my indebtedness in this, as in every other study that I
pursued under his direction, to my friend and teacher,
Professor Henry L. Eustis, of the Lawrence Scientific
School.   It was his cordial appreciation and sympathy
that first encouraged me to pursue the path of these
investigations.   This was twenty-five years ago, but I
have not forgotten it, and have borne in mind, as
every new point has presented itself, the pleasure he
would take in following my argument.

<div align="right">W. R. W.</div>

New York, Dec. 20, 1882.

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MODERN PERSPECTIVE.

CHAPTER I.

THE PHENOMENA OF PERSPECTIVE.

A DRAWING made *in perspective* undertakes to represent objects of the shape and size that they actually appear from a given point. It has to do only indirectly with their real shape and size, being mainly concerned with their apparent outlines and dimensions. Before trying to learn how to draw them, then, it is obviously desirable to find out how they really look. This first chapter will accordingly be taken up with considering the *appearances* of things, the *phenomena* with which perspective has to do.

The *things* in question, as always in the scientific study of form, are lines, especially straight lines; plane figures, especially rectangular figures and the circle; and solid objects, especially the sphere and cylinder. The appearance of solids bounded by plane surfaces is determined, of course, by the aspect of the plane figures that bound them.

1. Certain phenomena in regard to the shape and size of these things are sufficiently obvious. It does

not need to be pointed out that everything seems smaller — that is to say, subtends a smaller visual angle — when at a distance from the eye than when near ; that consequently the more distant portions of a straight line seem smaller than equal divisions near at hand ; that in rectangular figures the farther sides occupy less space to the eye than the nearer sides, so that they present, in most positions, a trapezoidal rather than a rectangular aspect, the sides inclining towards one another ; that a circle when seen in perspective generally appears as an ellipse, and that the centre of the circle does not occupy the centre of the ellipse, but is nearer to the farther than to the hither edge.   These *qualitative* determinations are easy enough.   But it is not so easy to determine the relations of *quantity*, to tell *how much* smaller a given distance will make a given line appear, or just at *what* angle the sides of the rectangle seem inclined, and in what direction they seem to run.   To determine these things with exactness is the chief object of these methods, — an object to be reached through the study of another class of phenomena, the appearances not of limited and finite lines and planes, but of lines and planes supposed to be indefinitely extended.   Indeed, finite lines and planes are in perspective considered merely as portions of the indefinitely extended lines and planes in which they lie.

2. The position of the spectator, that is to say, of the spectator's eye, is called the *Station Point.*

The station point.

3. All lines lying in one and the same direction, and consequently parallel to each other, are said to belong to the same *system* of lines. Each line is an *element* of the system. Systems of parallel lines.

Now if we imagine the lines of any system to be indefinitely extended both ways, we shall encounter the following phenomena.

4. All the lines of a system, that is, all lines parallel to each other in space, seem to converge towards two infinitely distant points. These points are called the *vanishing points* of that system of lines. They are 180° distant from each other. Vanishing points.

The vanishing points of a line are the utmost possible limits of its apparent extension, even though infinitely extended. For a straight line, although infinitely long, cannot subtend an arc of more than 180°; it cannot seem more than a semicircle.

The beams of the sun, or the shadows of clouds, at sunset, which seem to separate overhead and converge near the opposite horizon, afford a capital instance of parallel lines with two vanishing points. So also do parallel lines of cloud, and, in streets, the lines of sidewalks, eaves, and house-tops. They appear as great circles of the sphere of which the eye is the centre.

5. Now what is very curious is that whichever element of the system one looks at seems straight, the others, on both sides, seeming concave towards it. The horizon itself, which seems straight when one looks at it, seems curved if one looks up or down. Other hori-

zontal lines, when regarded with reference to the hori-
zon, seem parallel to it, and farthest removed from it,
where they are nearest the eye, approaching it at a con-
stantly increasing angle as they retreat towards their
vanishing points.

These singular phenomena, though constantly before
our eyes, are little noticed, and consequently but little
known ; but they sometimes force themselves upon
the draughtsman's attention, causing much confusion in
his drawing and in his mind. The fact that most
straight lines, all indeed except one, always seem
curved is the basis of the method of curvilinear or
panoramic perspective, which will form the subject of a
subsequent chapter.

6. Either vanishing point of any system of lines may
be found by. *looking* in the direction followed by the
lines of that system ; the vanishing point will then be
seen full in front of the eye.

That element of the system which passes through
the eye, or station point, will be seen *endwise*, the
line appearing as a point, coinciding with and cover-
ing the vanishing point, which is at its farther ex-
tremity.

7. In like manner, all planes parallel to one another,
and whose *axes* accordingly belong to the same system
of lines, are said to belong to the same *system* of planes.
Each plane is an *element* of the system. By the *axis* of
a plane is meant any line at right angles, or perpendic-
ular, to it.

8. All the planes of a system, that is, all planes par-
allel to each other in space, seem to converge Systems of
towards an infinitely distant line, which is parallel
planes.
the limit of their utmost extension. A plane, though
seemingly infinitely extended, like the sea, cannot sub-
tend an arc of more than 180° in every direction; it
cannot seem more than a hemisphere. Its limiting line
accordingly will be a great circle of the in- Vanishing
lines, *traces*,
finite sphere, of which the eye, or station point, or horizons.
is the centre. This line is called the *horizon*, or *vanish-
ing trace* of the system of planes.

9. The vanishing trace or horizon of any system of
planes may be found by glancing along that plane of
the system which passes through the eye. On looking
in any direction at right angles to the axis of the
system of planes, it is seen full in front of the eye.
That element of the system of planes which passes
through the eye is seen *edgewise*, the plane appearing
as a line, covering and coinciding with the vanishing
trace, or horizon of the system, which is its outer
extremity.

Such a line is *the* Horizon, which limits at once the
plane of the earth, and the plane, or hemisphere, of the
sky. We might call such a line the *vanishing line* of
a system of planes, just as we speak of the vanishing
point of a system of lines; but as it is common to call
any indefinitely extended right line a vanishing line, it
is more convenient to borrow a term from Descriptive
Geometry, and speak of the *vanishing trace* of a system
of planes; or we may borrow another word, and speak

of the *horizon* of a system of planes, distinguishing the real Horizon, or vanishing trace of horizontal planes, by a capital H.

10. Any point or line lying in a right line passing through the eye seems exactly to cover and coincide with the vanishing point of the system to which the line belongs. So also any line, figure, or surface, lying in a plane passing through the eye, appears as a right line, and seems to cover and coincide with a portion of the trace of the system to which the plane belongs.

11. Vanishing points and vanishing traces, or horizons, have to do only with the *direction* of lines and planes, not with their *position*. Hence, objects whose lines and planes are parallel have the same vanishing points and traces, whatever their position to the right or to the left, above or below the spectator.

12. A plane surface upon a solid object cannot be Solid objects. seen unless it is on the side of the object towards the trace of that plane.

13. It is obvious that all systems of horizontal lines have their vanishing points in the Horizon, and conversely, that the Horizon passes through the vanishing points of all systems of horizontal lines. The same is true, of course, of vertical or inclined planes, and the lines that lie in them or are parallel to them. From these considerations we can frame the following propositions, which are the fundamental propositions of our system of perspective.

(*a*) All lines, or systems of lines, lying in or parallel to a system of planes, have their vanishing points in the trace or horizon of that system. Hence : —

The three principal Rules of perspective.

I. *A line lying in a plane has its vanishing point in the trace of that plane.*

Conversely : —

The trace, or horizon, of any system of planes passes through the vanishing points of all lines parallel to them.   Hence : —

II. *The trace of a plane passes through the vanishing points of any two lines that lie in it,* that is, of any two elements of the plane.

(*b*) The traces of all the systems of planes which can be passed through a line, or parallel to it, in any direction, pass through the vanishing point of the system to which the line belongs, and intersect each other at that point.

Conversely : —

A line, or system of lines, lying in or parallel to two planes, has its vanishing point at the intersection of their traces.   Hence : —

III. *The line of intersection of two planes has its vanishing point at the intersection of their traces.*

14. The reader is recommended to take the pains not only to satisfy himself of the truth of these propositions, which he will easily do, but also to verify them by examples, determining for himself, in his daily walks, at what distant points in the earth or the sky the vanish-

ing points of different lines are to be *looked* for, lines
horizontal, vertical, or inclined; and in like manner to
trace the *horizons* of the different planes he encounters
in roofs or walls, exemplifying these propositions over
and over again until they become perfectly obvious and
familiar.

The vanishing points of the eaves, for example, and of
the raking cornice or other steepest line of a roof, are
easily found by *looking* in the directions they pursue.
These two directions determine the inclination of the
plane of the roof in which they lie. Its trace, or *horizon,*
is a great circle, or straight line, cutting across the sky
from one of these vanishing points to the other. In the
case of two intersecting roofs, the vanishing point of the
hip or valley that marks their intersection is found at
the intersection of their horizons.

15. The discussion of a problem in perspective can-
not be considered complete until the vanishing point of
every line and the vanishing trace of every plane has
been determined.

CHAPTER II.

PHENOMENA RELATING TO THE PICTURE.

IN the first chapter we considered the phenomena of perspective in nature; that is to say, certain appearances of the geometrical lines and surfaces with which perspective has to do.

Let us now — deferring to the fifth chapter all question of exact magnitudes, and of the precise determination of forms — consider in like manner the principal phenomena, the main characteristics, of a perspective drawing.

In so doing we will leave all *quantitative* determinations till by and by, and assume, or guess at, any dimensions or other data we may need; or determine them by judgment, or by the eye. But as this is just the way that such data are always determined when sketching, either from nature or from the imagination, it follows that what we have now to say is specially interesting to the artist and the amateur, since it comprises almost everything that he needs in his own work.

16. The picture is supposed to be drawn upon a plane surface called the *plane of the picture*, and so drawn that if the picture were transparent, The plane of the picture.

every point and line of the drawing would cover and coincide with the corresponding points and lines of the objects represented, as seen from a given position, the Station Point; the plane of the picture being at a given distance and in a given direction.

17. The distance and direction of the picture are taken upon a line passing through the station point, at right angles with the plane of the picture. This line is of course an axis of that plane (4). It is called *the* Axis. If the Axis is horizontal, the picture is vertical, and this is the usual position. But if the Axis is inclined to the horizontal plane, the plane of the picture is at an angle with the vertical direction, as sometimes happens.

18. The point in the plane of the picture nearest to the eye, or Station Point, S, is called the Centre of the picture, C. It is the point where the Axis pierces it. The distance from the Station Point to the Centre is the length of the Axis.

The Axis, Centre, and Station Point. Perspectives.

The term Point of Sight is used sometimes to designate the Station Point, sometimes the Centre of the Picture. Since it is thus ambiguous, we will not employ it.

19. The representation in a perspective drawing of a point, or line, or of the vanishing point of a line or system of lines, or of the vanishing trace, or horizon, of a plane or system of planes, is called the *perspective* of the point, or line, or vanishing point, or trace.

Figure 1 represents the picture-plane, PP, as a transparent plane on which are drawn the perspectives of the lines behind it. The perspectives of the lines drawn on the vertical plane behind it, and which are consequently parallel to the plane of the picture, are parallel to the lines themselves, whether horizontal, vertical, or inclined, and though shorter, they are divided proportionally to them. The perspectives of the lines at right angles to the plane of the picture, however, differ from them in both magnitude and direction. This illustrates the following propositions.
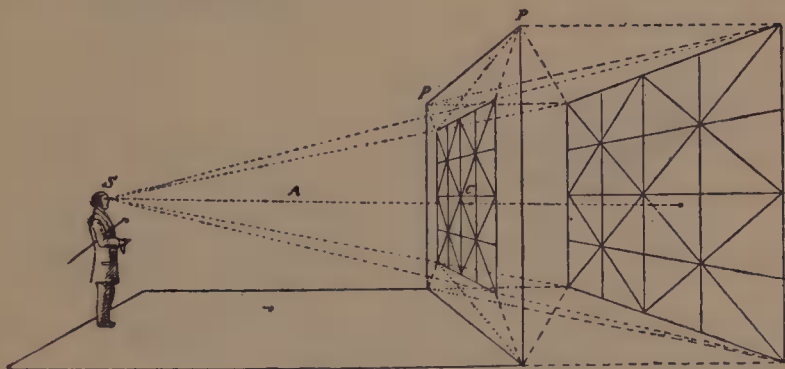
Fig. 1.

20. Lines parallel to the picture-plane, whatever their direction, have their perspectives drawn Lines parallel to the picture. parallel to themselves; that is, in their real direction. The magnitude of the perspective of any such line is less than that of the real line, according as the distance of the line itself from the picture is greater, but its parts are proportional to the corresponding parts of the line represented.

21. When such lines are parallel they have their

perspectives parallel to each other, and to the lines themselves.

22. The perspectives of lines not parallel to the plane
Lines inclined   of the picture are not parallel to the lines
to the picture. themselves, nor to each other, but are drawn
converging towards a point which is the perspective of their vanishing point. In this case, as the real lines *seem* to converge towards their real vanishing point, so their perspective representations *do* converge towards the perspective of their vanishing point.

23. Hence if the picture-plane be vertical, as it
Horizontal   usually is, the perspective of the horizontal
lines.   lines that are *parallel to the plane of the pic-ture* will be horizontal and parallel, that of inclined lines parallel to the picture will be parallel and inclined at the same angle with the lines themselves, and that of all vertical lines will be drawn vertical and parallel. All other systems of lines, whether horizontal or inclined, will have their perspectives converging to the perspective of their vanishing points.

24. Of the two vanishing points belonging to every system of lines (4) one will, in general, be behind the spectator, and one in front of him ; this last will be behind the plane of the picture, and its perspective will be somewhere in the plane of the picture and at a finite distance. But if the lines of the system in question be parallel to the plane of the picture, the perspective of both vanishing points will be at an infinite distance upon it, in opposite directions ; and lines drawn to them will, of course, be parallel (20).

25. It is often asked why the apparent convergence of vertical lines is not represented by the con- Vertical vergence of their perspectives, just as much lines. as that of horizontal lines. It is, just as much. For it is only those horizontal lines *which are inclined to the picture* whose perspectives are drawn to a vanishing point. The perspectives of horizontal lines parallel to the picture are drawn parallel to themselves, just as those of vertical lines are. And when the Axis is inclined so that the plane of the picture is no longer vertical, and vertical lines are no longer parallel to it, they, too, are drawn converging, one of their vanishing points, either the zenith or nadir, being now behind the picture, and its perspective at a finite distance upon it. This case will be discussed hereafter. See Plate VII.

26. Moreover, although lines parallel to the picture, whether vertical, horizontal, or inclined, do seem to converge towards a distant vanishing point just as other lines do, it is not necessary to represent this convergence, since their perspective representations in the plane of the picture also seem to converge as they recede from the eye, and in the same degree, covering and coinciding with them. The perspectives of the lines are themselves foreshortened, and the space between them diminished by distance.

27. To obtain this effect, however, in due degree, as, indeed, to obtain the just value of all other perspective effects, the eye of the spectator must remain at the station point. From other points the picture necessarily

looks inexact or distorted. These distortions increase
from the centre outward; and since it is so inconvenient
as to be practically impossible to keep the eye always at
the station point, it is best, in order to keep this distor-
tion within reasonable limits, not to extend the picture
more than 60°, *i. e.*, not to make it much wider than its
distance from the eye.

Some other phenomena relating to perspective draw-
ings are represented in Plate I., Figure 2.

In this plate, though a variety of objects are indicated,
only one direction of each kind is employed. All the
Plate I.   right-hand horizontal lines belong to a single
system, and all the left-hand lines to another. This, of
course, would not happen to this extent in nature; but
we have imagined, for simplicity's sake, that in the scene
represented all the buildings are parallel, and that all
the roofs are of the same pitch.

The Centre of the picture, C, the point nearest the
eye and opposite the station point, S, is here not exactly
in the middle of the picture, but considerably to the
right, being just below the church on the hill. The
Station Point is about six inches from the paper; and
the eye must of course occupy this position in order to
make the things represented appear of their proper
shape and size.

28. In this plate the following notation is adopted,
Notation.   a notation that will be adhered to throughout
this treatise, and with which the reader should make him-
self perfectly familiar. Each direction is indicated by a

single letter, the direction of each system of planes by
two letters, which give the direction of two elements
of the system ; each vanishing point by the letter V,
with the letter denoting the direction of the lines to
which it belongs written after it ; and the trace of each
set of planes by the letter T, and the letters denoting
the plane.

LINES AND THEIR VANISHING POINTS.

Lines.	Their Direction.		Their Vanishing Points.	
Z.	Vertical. (To the zenith, or nadir.)	[V^z]	Vanishing point of vertical lines.	
R.	Right-hand horizontal	V^ᴿ	" " Right hor. lines.	
	[*i. e.,* horizontal lines going off to the right.]			
L.	Left-hand horizontal.	V^ʟ	" " Left " "	
M.	Right-hand inclined upwards.	V^ᴹ	" " Right incl. "	
M′.	" " " downwards.	V^ᴹ′	■ " " " "	
N.	Left " " upwards.	V^ɴ	" " Left incl. "	
N′.	" " " downwards.	V^ɴ′	" " " " "	
P, Q }	{ Inclined lines formed by the	V^ᴾ, V^ᵠ }	" " { Lines of	
P′, Q′ }	{ intersection of inclined planes.	V^ᴾ′, V^ᵠ′ }	" " { Intersection.	

If there are several lines having the same general in-
clination they may be distinguished by figures, as R^1, R^2,
R^3, etc. Special vanishing points may be indicated as
V^1, V^2, V^3, etc.

PLANES AND THEIR VANISHING TRACES (OR HORIZONS).

Planes.	Their Direction.	Their Vanishing Traces.
RZ.	[Any plane of the system which contains, or is parallel to R and Z, i. e., right-hand vertical planes.]	TRZ. Trace of the right-hand vertical planes.
LZ.	To L and Z, *i. e.*, left-hand vertical planes.	TLZ. Trace of the left-hand vertical planes.
RL.	To R and L, *i. e.*, horizontal planes.	TRL. Trace of horizontal planes (*i. e.*, THE HORIZON).
RN.	To R and N, " Inclined up to the left.	TRN. Trace of the planes RN.
RN′.	To R and N′ " " down " "	TRN′. " " " RN′.
LM.	To L and M. " " up " right.	TLM. " " ■ LM.
LM′.	To L and M′. " " down " "	TLM′. " " " LM′.

29. The position of the various vanishing points, as well as the dimensions of the various objects, are supposed to be obtained, in this picture, as they would be obtained in a sketch from nature, or from the imagination.

If we place the eye at the station point and *look* in the direction followed by a system of lines, we shall see their vanishing point (8) ; and if the plane of the picture is interposed we shall see the perspective of the vanishing point in the same direction, covering the real vanishing point. Hence the perspective of the vanishing point of any system of lines is found by passing through the station point an element of that system. The point where it pierces the picture-plane is the perspective of the vanishing point of the system, coinciding with and covering the real vanishing point.

To find the Vanishing Point of a system of lines.

30. As M and M' are equally inclined to the horizontal plane, one looks up towards V^M at exactly the same angle that he looks down towards $V^{M'}$. V^M is accordingly just as far above the Horizon as $V^{M'}$ is below it.

The same is true of V^N and $V^{N'}$. But although M and N make the same angle with the ground, the distance of their vanishing points above the Horizon is not the same. For the eye at S, six inches in front of C, is further from V^L than from V^R, L being less inclined to the plane of the picture than R is. Looking up, then at the same angle, though the eye sees the real left hand upper vanishing point at the same height as the right-hand one above the real Horizon, it sees its perspective, V^N, higher up on the paper.

31. In like manner the perspective of the trace of a plane, or of a system of planes, is found by passing through the station point an element of the system. The line where it intersects the plane of the picture is the perspective of the trace, or horizon, of the system of planes.

For if the eye is at the Station Point, and glances along the element of the system passing through it, it will see the trace upon the plane of the picture, covering and coinciding with the distant horizon. The trace is the line where the plane passing through the eye cuts the plane of the picture.

32. And as the horizon of a system of planes passes through the vanishing points of all the lines that lie in it or are parallel to it, so does the perspective of the horizon, or trace, pass through the perspective of their vanishing points; and as all lines lying in or parallel to the system have their vanishing points in this horizon, so do the perspectives of all such lines have their point of convergence or vanishing point in this trace.

33. The propositions I., II., and III. (13), are thus as true for the perspectives in the plane of the picture as for the real lines and planes, vanishing points and horizons, as is exemplified over and over again in this plate. All the traces shown pass through several vanishing points, and every vanishing point lies in the trace of some system of planes. Every line which lies in two planes, as most of these lines do, has its vanishing point in both traces, that is, at their intersection; and the

traces of all the planes parallel to any one of these lines
intersect each other at its vanishing point.

34. It is specially to be noted that the lines of the
Lines of In-   hips and valleys lying at the intersection of
tersection.   two planes of the roofs have their vanishing
points at the intersection of the traces of these planes.

Thus the lines P, P′, Q, and Q′, being at the intersec-
tion, respectively, of RN and LM, RN′ and LM′, RN
and LM′, and RN′ and LM, we have

V^P at the intersection of TRN and TLM.

$V^{P'}$ "   "   TRN′ " TLM′.

V^Q "   "   TRN " TLM′.

$V^{Q'}$ "   "   TRN′ " TLM.

V^Q, the vanishing point of Q, is off the paper, being at
the intersection of the traces TRN and TLM′; and so in
like manner is that of Q′ at the intersection of TRN′
and TLM.

On most of the roofs the planes LM′ and RN′, being
on the further side, are out of sight. But the roof in
the extreme foreground shows all four slopes. It has
accordingly been selected for lettering.

35. This plate shows also that if the picture is verti-
Vertical   cal the trace of a vertical plane, such as LZ
planes.   or RZ, is a vertical line. For it must pass
through the vanishing point, V^z, of the vertical lines
that lie in it, and this point is the infinitely distant
zenith.

Besides, it is the line in which that plane of the sys-
tem which passes through the eye intersects the plane
of the picture (31); and as both these planes are verti-
cal, their intersection must be vertical.

36. Hence the hips and valleys, P and P′, which lie in parallel vertical planes, and accordingly have their vanishing points V^P and $V^{P\prime}$ in the trace of the system to which those planes belong, lie in a vertical line, one exactly above the other. As P and P′ are equally inclined to the horizontal plane, V^P and $V^{P\prime}$ as well as V^M and $V^{M\prime}$, or V^N and $V^{N\prime}$, are equally distant from the Horizon. These relations are indeed sufficiently obvious from the symmetry of the figure.

The vertical trace TPP′ is not shown in the plate.

37. The proposition that all lines lying in a plane have their vanishing points somewhere in the trace of that plane, receives special illustration in the case of the paths which cross the flat open space beyond the railroad; being level, they have their vanishing points on the horizon, one at V^R, others at V_3 and V_2. The ladder lying on the roof to the right has the vanishing point of its sides, which are supposed to be made parallel, at V^1 in the trace of the plane of the roof, TRN′.

This proposition is very serviceable in putting in any parallel lines on any plane, as, for instance, in drawing the diagonal lines of slating on the roof to the left, the vanishing points being shown at V_4 and V_5 in the trace of RN, the plane of the roof.

38. But if a line lying in a given plane is parallel to the plane of the picture, then its vanishing point, though still in the trace of the plane, will be at an infinite distance; and the line itself, being still directed to its vanishing point, will be parallel to the trace. For lines which

Lines parallel to the picture parallel to the traces of the planes they lie in.

meet only at an infinite distance are parallel. Hence
we have the general proposition that, in any plane, a
line which is parallel to the picture will be drawn
parallel to the trace of the plane.

39. In the figure the broken line $p\,p\,p$ is drawn paral-
Sections tak- lel to the plane of the picture. It is the line
en parallel to
the picture. of intersection of a vertical plane parallel to
the picture, cutting across the front corner of the princi-
pal building. In each plane it is parallel to the trace
of that plane.

40. That this must be so follows not only from 38,
but also from the general proposition, that, if one sys-
tem of parallel planes intersects another system, their
lines of intersection are all parallel.

For a line lying in any plane, and parallel to the
plane of the picture, may be regarded as the intersection
of that plane by a plane parallel to the picture. But the
trace of the system of planes in which the line lies is the
line in which a plane parallel to those planes and pass-
ing through the eye intersects the plane of the picture.
We have thus two inclined planes parallel to each other,
intersecting two vertical planes parallel to each other.
Their intersections are accordingly parallel, and the line
in question is parallel to the trace of the inclined plane
in which it lies; and since it is parallel to the picture,
its perspective is parallel to itself, and also is parallel to
the trace: $Q.\ E.\ D.$

41. Moreover, if any plane of that system of planes is
The intersec- extended so as to cut the plane of the pic-
tion of any
plane with ture, that intersection is also parallel to the
the plane of
the picture. others, and to the trace in question.

42. This plate illustrates also the proposition (12) that in the case of solid objects the plane surfaces by which they are bounded are visible only on the Planes visible only sides towards their traces, or horizons (12); when below their horizons they are visible only when they are, so to zons. speak, *below* their horizons. We see in the plate that the roof most nearly below the eye, being below all the horizons, shows all its slopes, and so does the next one to the left. In all the others, one, two, or three planes disappear, as they are above their traces; until, at last, in the case of the church on the top of the hill, which is above all the traces, all the roofs are out of sight. One of the houses in the fort on the lower hill shows the roof LM just disappearing, the lines L and M both co-inciding with the horizon TLM.

It is to be observed that these traces, being portions of great circles, do not terminate at the vanishing points by which their position is determined, but pass through and beyond them.

43. The perspective of a line or of a point is often called a perspective line or point, or when speaking of the picture, simply a line or point. But in this last case, to avoid confusion of mind, one must be careful to notice whether the real line or point and the in-finitely distant vanishing point are spoken of, or their representations in the plane of the picture.

So in speaking of the trace or horizon of a system of planes, one must be careful to notice whether the infinitely distant line, or the line in the picture

which covers, and to the eye coincides with it, is in-
tended.   It might be well, for distinction's sake, to call
the former, or distant imaginary line, the horizon of
the system of planes ; and the latter, or line where the
plane passing through the eye cuts the picture, its
trace.

CHAPTER III.

SKETCHING IN PERSPECTIVE.   THE PERSPECTIVE PLAN.
THE DIVISION OF LINES BY DIAGONALS.

HAVING in the first chapter considered the nature of the phenomena with which perspective drawings have to do, we examined in the last chapter the aspect of the drawings themselves, first observing the relation which lines parallel to the plane of the picture bear to their perspective representations, and then, in the case of those not parallel to the plane of the .picture, the relation that the perspective lines and planes, by which the objects represented are defined, bear to the perspective of their vanishing points and vanishing traces.

Plate II. illustrates almost all the points raised in explaining Plate I.; the roofs that are be- Plate II. low their traces being all visible, and those that are above them being all out of sight, while all the lines of intersection of the planes converge to the intersection of their traces.   This is specially noticeable of the *valleys* of the main roof in the lower picture.

The trace T P P' of the system of vertical planes P P to which these valleys are parallel, and which accordingly passes through their vanishing points (13, II.), and which was not drawn in the previous plate, is here shown.

Since the lines which indicate the position of these valleys in the perspective plan lie in the same vertical planes with the valleys themselves, they must have their vanishing point in this same vertical trace T P P' ; and since, like all the lines of this plan, they lie in a horizontal plane, their vanishing point must lie in the Horizon (13, I.) : it is therefore to be sought at the intersection of T P P' with the Horizon, at the point marked V^X (13, III.). Since the vertical planes in which these valleys lie are obviously at an angle of 45° with the principal vertical planes R Z and L Z, this line X, whose vanishing point is at V^X, is at 45° with the lines R and L. V^R is off the paper.

44. If we put the eye at the station point S, four or The Vanishing Point of 45°. five inches in front of C, and, looking first at V^R and V^L in directions at right angles to each other, look then between them so as exactly to divide the angle, we shall be looking in the direction X, and shall see V^X directly in front of the eye (6).

We will retain this notation, V^X, throughout this treatise, to denote the vanishing point of horizontal lines making an angle of 45° with the principal horizontal lines, R and L ; and shall call it, for brevity, the vanishing point of 45°.

The little building on the left has steeper roofs than the other, their slope being the same as that of the roof of the tower. Their vanishing points are accordingly V^{NI}, V^{MI}, $V^{NI\prime}$, etc., which are off the paper.

As the tower roof is supposed to slope alike all round, the hips P_1 and P_1' lie also in parallel planes, at 45° ;

their projection on the perspective plan has V^x for its vanishing point, and V^{p1} and $V^{p1\prime}$ lie in the trace of the plane $P P\prime$, at equal distances above and below the Horizon (36).

45. The position of these vanishing points and traces is supposed to be determined just as the position of the other leading points in the picture is determined; that is to say, their relative position on the paper is made to correspond to the relative position of the real points and vanishing points, as nearly as may be, by the eye, by looking first at the point, and then looking for the corresponding place on the paper. The position of the leading vanishing points being thus determined, the vanishing traces can be drawn connecting them, and new vanishing points, such as V^p and $V^{p\prime}$, determined by their intersection.

If the propositions illustrated in the last chapter are borne in mind, a consistent and tolerably cor- Sketching. rect perspective sketch can easily be made, the eye being greatly aided in its estimate of the relations of things, and their apparent shape and dimensions, by the considerations to which attention is thus directed. The principal points being fixed by the eye, the other points are then determined, partly by the eye, partly by means of lines drawn to the vanishing points.

46. A great advantage may also be found in the use of a *perspective plan* of any object that is to be The Perspective Plan. drawn, especially in sketching, not from ob-

jects but from the imagination.   Thus in the figure
Figure 5.   (Fig. 5), although the main building could be
drawn without much chance of error, it is by no means
so easy to determine just where the tower behind it
should make its appearance over the roof.   By complet-
ing the plan, however, as is done by dotted lines, its
position is at once determined.   The objection that it is
undesirable to cover the drawing with construction lines
may be entirely met by drawing the plan at a lower
level, as if it were the plan of the bottom of the cellar,
ten or twenty feet underground ; and for the purpose in
hand, the cellar may be supposed to be of any conven-
ient depth, so as to get the plan entirely out of the
picture, as is done in the figure.

47. This sinking of the perspective plan has two inci-
dental advantages.   In the first place, it makes it prac-
ticable to draw it on a separate piece of paper, which
may be removed and kept for use a second time, if, as
often happens, a perspective drawing needs to be made
over again.   In the second place, it defines the positions
of things much more accurately ; the lines by whose
intersection the position of the vertical lines is deter-
mined cutting each other more nearly at right angles.
It will be seen in the figure that the lines in the real
floor-plan cut each other so obliquely that it is not easy
to tell exactly where the corners of the tower do come.

48. It follows from this that the level at which the
Bird's-eye   object is to be shown in perspective is quite
Views.   independent of the level chosen for its plan.
This also is illustrated in Plate II., the same plan

serving for three representations of the building, at different levels, — one nearly even with the eye, with a bird's-eye view below, and with what might be called a *toad's-eye* view of it above. The same vanishing points being employed in all three sketches, the phenomena pointed out in the previous chapter, of the appearance and disappearance of plane surfaces according as they come below or above their horizons, are here again illustrated.

49. In thus sketching in perspective, whether from nature — that is, from a real object — or from the imagination, it will be found much easier to determine vertical magnitudes than horizontal ones ; that is to say, it is easy to determine the position of horizontal lines, but not their length ; and the length of vertical lines, but not their position.

In the sketch, for example, the position of the vanishing points, and the position and height of the front corner of the building to be represented, being once assumed or determined, other heights, whether equal or different, can easily be determined by means of parallel lines drawn to the vanishing points. The height of an object having been assumed in one part of the picture, an object of the same height can be put in anywhere else by the employment of parallel lines.

But though it is thus easy to represent the three gable-ends in this sketch as being of the same height, it is not so obvious how to draw them so that they shall all seem equally wide.

50. Moreover, the subdivision of the perspective of

The Division of Lines. vertical lines, whether into equal parts or according to some given proportion, presents no difficulty ; for the vertical lines are parallel to the picture, and their perspectives will accordingly be divided just as the lines themselves are (20).

But while the division of vertical lines and their apparent diminution in size is easily managed, the subdivision of horizontal and inclined lines (except those which like the vertical lines are parallel to the plane of the picture) is a matter of difficulty. The more remote divisions are smaller, but it is not clear how much smaller.

Two methods are adopted to determine this, — the method of Diagonals, and the method of Triangles. Let us take the first, first.

The method of Diagonals is illustrated in the various

The Method of Diagonals. figures of Plate II. It applies to parallelograms whose perspectives are given or assumed the following propositions : —

51. *Proposition* 1. A line drawn through the intersec-

Halving, quartering, etc. tion of the diagonals of a parallelogram, parallel to two of its sides, bisects the other sides and the parallelogram itself.

This process may be repeated with each half, and the

Fig. 3, a. given figure, or any line in it, divided into 2, 4, 8, 16, or 32 equal parts, etc. See Fig. 3, *a*.

The application of this to the perspective of a paral-

Fig. 5. lelogram is shown in Fig. 5, where the left-hand side of the larger building is thus divided.

52. Less familiar is the employment of this principle to ascertain the vertical axis of a tower two of whose sides are given in perspective, as in Fig. 5. If diagonals are drawn across the tower, from two points on the right-hand vertical corner to points at the same levels on the left-hand corner, they will intersect in the middle of the tower, and a vertical line through their intersection may be used to determine the apex of the roof which covers it, as in the figure. These diagonals lie in a vertical plane that crosses the tower diagonally.

53. This is the common way of dividing a perspective line or surface into halves; and it is constantly used, as on the left-hand side of this building, and on the right-hand side of the building above (Fig. 4), to determine Fig. 4. the centre line of a gable, and the position of its apex.

54. It is obvious that this furnishes an alternative method of determining the slope of these roofs. Instead, that is, of fixing the position of the vanishing points of M and M′, P and P′, and thus obtaining the direction of these inclined lines, we may assume at once the direction of any one of these lines, say the nearest one. The intersection of this line with the central vertical line fixes the height of the roof; the other slope and the other roofs are then easily drawn.

55. Perspective is full of these alternative methods, different ways of doing the same thing. Which way it is best to adopt in any given case, depends upon the nature of the case. In the present instance, the vanishing points V^M and $V^{M\prime}$ being outside the picture, the method of diagonals is rather the most convenient.

56. It is to be observed, however, that though V^M and V^{M1} are off the paper, V^F and V^{P1} are within easy reach. It is generally worth while, accordingly, to fix the position of the more remote vanishing points, so as to determine the position of the traces or horizons that lie between them, and of the points where those traces intersect, even if we make no direct use of the vanishing points themselves. Thus, in the plate, although the points V^{M1}, V^{N1}, $V^{M1\prime}$, and $V^{N1\prime}$, which give the slope of the roofs of the small house and of the tower, are all at a distance, the traces of the planes of the roofs TRN_1, TLM_1, $TRN_1\prime$, and $TLM_1\prime$ all cross the paper, and their intersections V^{P1} and $V^{P1\prime}$ are close at hand.

57. *Proposition* 2. If through the intersection of the diagonals a second line is drawn parallel to the other two sides of the parallelogram, a single diagonal suffices to effect the subsequent subdivisions, as is exemplified in Fig 3, *b*, and on the left-hand side of the larger building in Fig. 6 below.

Fig. 3, *b*.
Fig. 6.

58. *Proposition* 3. *Conversely,* if a line drawn from one corner of a parallelogram to the middle of one of the opposite sides be continued until it meets the other side, prolonged, the length of that side, or of the parallelogram itself, may be doubled, and by a repetition of the process, tripled, quadrupled, etc. See Fig. 3, *c*.

Doubling, tripling, etc.

Fig. 3, *c*.

This proposition is of great use in perspective drawing, as may be seen in Fig. 5, where the gabled end on the right is several times repeated, each time smaller than before.

It will be seen that the gable ends of the roofs grow steeper and steeper, their lines converging, in fact, to the distant vanishing points M and M'. By obtaining those points, the accuracy of these results can be tested.

59. *Proposition* 4. If one side of a parallelogram be divided in any way at one end, equal divisions may be laid off at the other end by means of two diagonals. See Fig. 3, *d*. Symmetrical division. Fig. 3, *d*.

This is very useful in giving a symmetrical treatment to a surface shown in perspective, as is seen in the left-hand building, Fig. 5. The position and width of the nearer window on the side of the building being assumed, the vertical lines enclosing the further window are easily found.

At the end of the building the inclined lines of the gable, which may be regarded as the semi-diagonals of an unfinished parallelogram, answer the same purpose. The base of any isosceles triangle can be divided in this way.

60. *Proposition* 5. If one side of a parallelogram be divided in any way, the adjacent sides may be similarly divided into proportional parts, by means of one diagonal; and by using the other diagonals the order of the parts may be reversed. See Fig. 3, *e*. Proportional division. Fig. 3, *e*.

By this means any required division of a line given in perspective may be effected, as is shown in Fig.

6, on the right-hand sides of both buildings. The re-
quired division is made on the vertical line, and then
transferred to the horizontal line by means of the diago-
nal, the nearest corner of the small house being divided
according to the desired position of the door and win-
dows, and that of the large building into three equal
parts.

61. If the diagonal makes an angle of 45° with the
adjacent sides, their segments will of course be not only
proportional, but equal, each to each.

In the perspective plan of the small building, for ex-
ample, in which the diagonals are directed to V^X, the
" vanishing point of 45°," and accordingly make an
angle of 45° with the sides of the building, it appears
that the window is just as far from the corner on one
side as the farther edge of the door is on the other. It
appears also that the plan of this building is just four
squares, though it hardly looks so, the side being greatly
foreshortened, while the main part of the other building
is just as broad as it is long, comprising nine squares,
each as large in plan as the tower.

62. In applying this proposition to a perspective
drawing, the line on which these parts are first laid off
must of course be a vertical line, or some other line
parallel to the plane of the picture, as it is only in the
case of such lines that the division of the perspectives is
proportional to that of the lines themselves (20).

63. *Proposition* 6. It is not necessary that the length
of this line shall be previously determined. Indeed,

it is more convenient that it should not be, as it is easier
to establish a given ratio of parts on an indefinite line.
The equal or proportionate parts may be set off at any
convenient scale, on any convenient line that touches
the end of the line to be divided, and the
diagonal drawn without completing the paral-   Fig. 3, *f.*
lelogram, as in Fig. 3, *f.*

The division of the long wall in Fig. 6, for instance,
is effected by setting off three equal distances upon the
further corner, just as well as by dividing the near corner
into three equal parts.

64. It is not necessary in any of these cases, of course,
that the parallelogram shall be a rectangle. The inclined
line N, for example, in the middle of the upper figure,
Fig. 4, is divided into four equal parts by equal
divisions laid off on the vertical line that bisects the
gable.

In these last propositions, it will be observed, use has
been made of only half a parallelogram, that is to say, of
a triangle.

65. Proposition 5 may then be restated as follows : —

If one side of a triangle be divided in any way, the
adjacent side may be divided into proportional parts by
means of lines drawn parallel to these two
sides and meeting on the third side. See *f,*   Fig. 3, *f.*
Fig. 3.

66. And from Proposition 6 we may derive this : —

If from one end of a line there be set up an auxiliary,
parallel to the plane of the picture, any parts taken upon

the perspective of this auxiliary may be transferred to the perspective of the line, in their true proportions, by means of a third line joining the last point taken on the auxiliary with the other end of the first line (65).

67. But as any line, drawn in the picture at random, *Random lines used for proportional division.* may be conceived of as being the perspective of a line, which it exactly covers and conceals, drawn parallel to the picture, it follows that *any line whatever, touching one end of a perspective line, may be used as an auxiliary by which to divide it in any required proportion ;* and, the triangle being completed, the first segments of the broken lines by which the proportions are transferred will be parallel to the line to be divided and will be directed to its vanishing point, and the second segments will be actually parallel to the auxiliary line, since its vanishing point is at an infinite distance.

Thus, if it is required to erect six equidistant spikes upon the ridge of the left-hand building, we may from either end of the ridge draw a line in any convenient direction, and lay off on that line five equal parts, using any convenient scale, as in the figure. Completing the triangle and proceeding as above, we get the points of division desired. This triangle does not lie in a vertical plane but in an oblique plane, containing both the horizontal line to be divided and the auxiliary line ; this line is parallel to the picture, and is shown in its true direction. Fig. 5, Fig. 6.

Finally, since any line whatever that touches the end of the line to be divided will serve this purpose, it is

often convenient, instead of drawing a new line, to employ a line already existing. We may, for example, as in Fig. 6, lay off our five equal parts on the sloping line of the further gable, and obtain the same points on the ridge as before. This line, however, is now conceived to represent, not the line of the gable, but a line parallel to the plane of the picture, so taken as exactly to cover the line of the gable, so that their perspectives coincide.

CHAPER IV.

THE DIVISION OF LINES BY THE METHOD OF TRIANGLES.

THE third chapter first set forth the convenience, in making a perspective drawing, of putting into perspective the plan of the object to be drawn, and of sinking this plan so far below the representation of the object as to get it quite free from the picture. Plate III., Fig. 10, affords further illustration of the use of the perspective plan. The plan of the gatehouse on the left is indeed below the picture, having in fact been drawn on another piece of paper, and removed, as suggested in a previous paragraph (47). But the plan of the one in the distance on the right is given, and it serves to determine all the principal horizontal dimensions. The plan of the principal building, the barn in the valley, is drawn above it, instead of below, as is sometimes most convenient, especially in high buildings, in the upper parts of which it is of advantage to have the perspective plan near at hand. It is often a convenience, also, to make several plans, set one above another, taken at different levels. In the plate, for example, we have first the plan of the outline of the walls, to determine the position of the doors and windows, and of the posts of the shed, and then just above it the plan of the

eaves, showing their projection, and the position of the brackets beneath them. As only the front part of the building is seen, only the front part of the plan needs to be drawn. In putting in the eaves, advantage is taken of the "vanishing point of 45°," V^X, to make them equally wide on each side.

In this plate the slope of the roofs and gables of this building, as well as of the smaller one with a hipped roof beyond it, is indicated by the same letters as in the previous plates, and their vanishing points accordingly by V^M, $V^{M'}$, etc., as before. The gables of the little gate-houses are so steep that their vanishing points are quite out of reach; and these gables are, in fact, drawn by the method of diagonals, as described in the previous chapter (54). The slope of the steps is given by V^{M1} and $V^{M1'}$, and the trace of the inclined planes of the bank by $T L M_1$ and $T L M_1'$. Their position shows that the banks are a little steeper than the roof of the barn. The diagonal braces of the fence have nearly the same slope as the barn shed, converging to points just below V^N and just above $V^{N'}$.

In this plate the centre, C, is again quite out of the middle of the picture. The station-point, S, the proper position of the eye, is about six inches in front of C.

68. The first of the two methods by which a line given in perspective may be divided up in any given proportion has already been described. It was shown that this, though called the Method of Diagonals, finally leads to the division of

<div style="text-align:right">The Method
of Diagonals.</div>

such a line by means of a triangle, one side of which is formed by the line to be divided, and one side by an auxiliary line, drawn parallel to the plane of the picture in any convenient direction and divided in the given proportion. The points of division are transferred from this auxiliary line first to the third side of the triangle, by lines parallel to the perspective line, and directed to its vanishing point; and then to the perspective line by lines actually parallel to the auxiliary.

69. Both these steps are obvious and simple applications of the proposition that lines drawn parallel to one side of a triangle divide the other two sides proportionally. But it does not yet appear what is the real direction of this third side, nor in what plane the triangle really lies; that is to say, the vanishing point of this line and the vanishing trace of this plane are not yet determined.

70. The other method of dividing perspective lines, called, *par excellence*, the Method of Triangles, is a more direct application of the same principle. The auxiliary line, as before, is drawn parallel to the plane of the picture; but the points by which it is divided are now transferred directly to the perspective line by lines drawn parallel to the third side of the triangle. Plate III. is devoted to the illustration of this method. Fig. 7, *a* and *b*, shows the difference between this method and the preceding. In each of the triangles here shown, the base is divided proportionally to the parts set off on the left-hand side. But in the upper ones the division is effected by the Method of Diagonals, as in Fig. 3, *f;* in the lower ones the

The Method of Triangles.

Fig. 7, a, b.

same result is reached, more directly and simply, by the Method of Triangles.

71. This application of the principle in question, however, though more direct and simple, is in one respect less easy of adaptation to lines given in perspective. For the two systems of parallel lines employed in the Method of Diagonals may be drawn without difficulty, the first having the same vanishing point as the line to be divided, and the second being actually parallel to the auxiliary line, since that line is parallel to the picture. But in the Method of Triangles, the lines by which the points are transferred are parallel to the third side of the triangle, whose vanishing point is not known. It is accordingly necessary first to find the vanishing point of this line.

72. This may be done at once, when, as in the plan of the eaves, at the top of the plate, the plane *Point and Line of Proportional Measures.* in which the auxiliary triangle lies is known; that is to say, when its trace or horizon has been already ascertained. The auxiliary line here lies in the horizontal plane, and the given line lies in the same *Horizontal lines.* plane; the whole triangle is accordingly in the horizontal plane, and all its lines have their vanishing points in the Horizon, — the given line at V^L, the auxiliary line at an infinite distance, and the third side of the triangle at V_1. This point is ascertained simply by prolonging this side until it reaches the Horizon (13 I.). If now it is desired to find the position of the ten brackets that support the eaves, it is easy to lay off on the auxiliary line nine equal divisions, and to complete the triangle :

by drawing lines parallel to the third side, the distances
set off on the auxiliary are at once transferred to the per-
spective line, the lines converging to V_1. This auxiliary
vanishing point is called the vanishing point of propor-
tional measures, or simply the point of measures. The
auxiliary line is called the line of proportional measures,
or simply the line of measures. These must not be con-
founded with the point and line of *equal* measures de-
scribed in the next chapter (98, 99, 125).

73. It makes no difference, of course, at which end of
the perspective line the line of proportional measures is
drawn, so that it is parallel to the picture. The relative
position of the doors, windows, etc., in the lower plan,
and of the posts of the shed, are laid off on lines
of measures drawn from their further ends, and the
points of division transferred to the lines of the plan
by means of the points of measures V_2 and V_3, both of
which, of course, are also on the Horizon. But it is ob-
viously conducive to precision to have the line of mea-
sures touch the nearer end of the line to be divided,
since, in general, converging lines give more accurate re-
sults than do lines of divergence.

Neither, of course, does the size of the proportional
parts laid off upon the line of measures affect the result.

Fig. 7, *c*.   In Fig. 7, *c*, the base of the triangle is divided
into the same four equal parts, whether the
parts taken on the adjacent side are large or small.
Any convenient scale may be used ; but that scale will
in general be found most convenient which makes the
line of measures about as long as the perspective line to

be divided, and which brings the point of measures within easy reach.

74. In the same way a line lying in a vertical plane may be divided by means of a vertical line of measures ; the point of measures or vanishing point of the third side of the triangle and of the lines drawn parallel to it being now in the trace of the vertical plane. If a line lies at the intersection of two planes, it is a mere matter of convenience whether the line of measures is taken in one plane or the other, or in which trace the point of measures is taken.

Vertical lines.

Thus the seven parts into which the length of the barn in Plate III. is divided may be taken either on a horizontal or on a vertical line ; that is to say, upon a line of measures parallel to the trace of either plane. The points at the bottom of the wall, on the left-hand side, which determine the position of the doors and windows, may be got either by means of a horizontal line of measures, as shown, with its point of measures on the Horizon, at V_4, or by a vertical line of measures, namely, the corner of the barn, on which the same proportional parts are laid off, with its point of measures on the trace of the plane L Z at V_5. Here the first triangle lies in the horizontal plane, and the second in the vertical plane, the first on the ground and the second in the side of the barn, as they seem to.

75. If a line lies in a plane inclined to the horizontal plane, as each inclined line of the gable-ends of the barn lies in the plane of its roof, a simi-

Inclined lines.

lar procedure may be followed. A line of measures may
be taken in that plane, touching the given line at one
end and parallel to the picture, the point of measures
being now in the trace of the plane of the roof.

76. And as in the horizontal plane a line parallel to
the picture is horizontal, and in vertical planes vertical,
— that is to say, in each case parallel to the trace of
the plane it lies in, — so in the case of an inclined plane,
a line lying in it parallel to the picture is parallel to the
trace of the system to which the plane belongs ; for
though its vanishing point is somewhere in that trace
(13, I.), it is at an infinite distance (20). A line directed
to that point is accordingly parallel to the trace (38).

77. The perspective of an inclined line can then be
divided in any required proportion, as easily as that of a
horizontal or vertical one, by drawing through one end
of it a line of proportional measures parallel to the trace
of the inclined plane in which it lies, and taking the
point of measures on that trace.

Thus in the plate the position of the brackets or pur-
lins on the gable of the barn is found by dividing each
slope into six parts, by means of a line of measures
drawn parallel to T L M, the trace of the roof in question ;
and as the sloping lines of the gable lie in a vertical plane
R Z, parallel to the side of the barn, as well as in the
plane of the roof, it follows that the position of the six
brackets can be found either by laying off equal parts, on
vertical lines, with points of measures on the trace of R Z,
at V_6 and V_7, or by laying off equal parts upon lines of
measures parallel to the traces of the planes of the

roofs, that is to say, parallel to T L M for the left-hand slope, and to T L M' for the other, with points of proportional measures at V_8 in T L M, and V_9 in T L M', respectively.

In the former case the triangles lie in the plane of the gable-end ; in the latter, each lies in the plane of its own roof.

78. It follows from the above, as has already been shown, that if any object bounded by plane surfaces be cut through by a plane parallel to the plane of the picture, the line of intersection on each face will be parallel to the trace of the plane in which it lies (39).   This is exemplified in the plate, where the dotted line, A A, running along the ground and over the barn, follows this law.   If the front corner of the building were sliced off parallel with the picture, this would be the line of the cut.   The same thing is exemplified on the front corner of the other building. *Sections parallel to the picture.*

We shall find use for this bye and bye, when we come to the perspective of shadows.

79. Finally, just as in the Method of Diagonals we found at last that the auxiliary line, or line of measures, may be taken in any direction, at random, so here the same thing is true.   For here too any line, drawn in any direction at random, from either end of a perspective line, may be regarded as the perspective of a line of measures, parallel to the picture and drawn parallel to the trace of the plane in which it lies.   This trace, then, will be parallel to it ; and since *Random lines of proportional measures.*

the plane contains the perspective line, its trace must
pass through the vanishing point of that line ; for the
trace of a plane passes through the vanishing points of
all the lines that lie in it (13 c) ; if then through
the vanishing point of the line we wish to divide,
we draw a line parallel to the assumed line of mea-
sures, we shall have the trace of a plane in which they
both lie ; and upon this trace the third line of the tri-
angle, joining the other end of the perspective line with
the last point taken on the line of proportional mea-
sures, will have its vanishing point. This point, the
point of measures, can be found, just as before, by pro-
longing the third line, the base of the triangle, till it
touches it.

80. The principle that the line of measures may be
drawn at random in any direction, the corresponding
point of measures being taken on a line or trace drawn
parallel to it through the vanishing point of the line to
be divided, is illustrated in the division into five equal
parts of the hip of the roof of the smaller building in the
middle distance. Here the line of measures is drawn
arbitrarily at about 60°, the auxiliary trace being drawn
through V^P, the vanishing point of the hip, and its point
of measures, V_{10}, determined on that trace.

The triangle here seems to lie in the plane of the
roof, but in fact it has nothing to do with it.

81. Moreover, since the only characteristic of this
auxiliary trace, relatively to the conditions of the prob-
lem, is this, that it passes through the vanishing point

of the line to be divided, it follows that any line drawn through the vanishing point of a given line may be regarded as the trace of a plane in which the given line lies, and will contain the point of measures corresponding to a line of measures drawn through either end of the given line parallel to it.

82. This gives us, in other words, this singular proposition : —

Of any two perspective lines having the same vanishing point, one may be taken as the trace of a plane passing through the other ; and if a third line be drawn parallel to the first, and touching one end of the second, any parts taken upon this third line may be transferred to the second in their true proportions by means of a point of measures taken upon the first. See Gwilt's Encyclopædia of Architecture, § 2457.

Perspective lines used as auxiliary horizons.

83. The position of the vertical bars of the cresting upon the ridge of the gate-house on the left is determined in this way, five equal parts being laid off upon a line drawn from the further end of the ridge parallel to the eaves of the roof, as a line of measures, and the point of measures, V_{11}, taken on the eaves.

The way in which the position of the vertical bars of the gate below is determined also illustrates this proposition. A line touching the top of the gate is drawn parallel to the ridge-pole, which has the same vanishing point, V^L. Equidistant points are taken on this line, and transferred to the top of the gate by a point of measures, V_{12}, taken on the ridge-pole. This reduces the

labor of dividing up a given perspective line in any required proportion to almost nothing.

84. Here, as in the corresponding case in the previous chapter, care is to be taken not to fancy that the line of measures, and the triangle determined by it, really lie in the plane they seem to lie in.

In this last case the triangle lies in an imaginary inclined plane, and is no more vertical, as it seems to be, than the point of measures is on the ridge, as it seems to be: it is really in the infinitely distant horizon which the ridge covers and coincides with.

Plate III. also furnishes illustrations of two points of general interest.

85. The first of these is the use of the point V^X, the vanishing point of horizontal lines making an angle of 45° with the principal directions R and L, to determine V^N, when V^M is given, the lines that slope up to the left being supposed to make the same angle with the ground as those that slope up to the right. If these inclinations are equal, the inclination of the planes R N and L M will be equal, as in the case of these roofs ; their lines of intersection, P, will lie in vertical planes, making 45° with the principal vertical planes ; their trace will be a vertical line passing through V^X, as shown in the previous chapter (42) ; and V^P will be at the intersection of this trace with T L M. If now T R N be drawn through V^R and V^P, V^N will be found at its intersection with T L Z, and $V^{N'}$ will be at an equal distance below.

The point V^{P1}, which determines the direction of the

line P_1, at the intersection of the two banks in the further corner of the barnyard, is found in like manner.

86. The second point is illustrated by Fig. 8, which shows how the true direction of the lines Q or Q', whose vanishing points are at the distant intersec- Fig. 8. tion of the nearly parallel traces T R N and T L M', or T R N' and T L M, may be obtained by means of two similar triangles, the common device for di- Fig. 9. recting a third line to the intersection of two given lines, as shown in Fig. 9.

This is applied in the plate, Fig. 10, to find the true direction of the left-hand line of the hipped Fig. 10. roof, just below the point C.

CHAPTER V.

ON THE EXACT DETERMINATION OF THE DIRECTION AND MAGNITUDE OF PERSPECTIVE LINES.

THE first two chapters were given to a general observation of the phenomena of perspective, in nature and in drawings, and the last two to an explanation of the practical making of such drawings, certain data being assumed. It was assumed that the position of the principal vanishing points, giving the direction of the principal lines, had been already determined, with more or less accuracy, by the eye or by the judgment, and that their length had also been fixed in the same way. The discussion showed how the position of other vanishing points and the length and direction of other lines could then be determined, and how any of the lines thus drawn could be divided up in any desired manner, that is to say, in any given proportion.

It is necessary, in order to conclude this part of the subject, to show how these data may be more precisely determined. It is necessary to show how, when the real direction of lines is exactly known, their vanishing points may be fixed with precision, and how, when their

real length is known, the exact length and position of
their perspective representations may be determined.
The position of the object to be drawn, the position of
the picture, and the position of the spectator's eye,
must, of course, be known also.

Plate IV. shows how these questions are answered:
Fig. 11 showing in plan, and upon a reduced Plate IV.
scale, the position of the spectator at the sta- Fig. 11.
tion point S; that of the picture at pp, which shows
the plane of the picture edgewise as it would appear
looking down upon it; and that of the object to be rep-
resented at A. Two elevations of this object, which is
a small house, showing its vertical dimensions, are
given alongside; the plan and the two elevations
together giving exact information as to the magni-
tude and direction of the lines defining it. The pic-
ture itself is shown between the plan of the house
and its own plan, just as if the plane of the picture,
pp, had been revolved backward into the plane of the
paper.

The object is here represented as being about six
times as far from the station point as the picture
is, the picture being about eighty feet from the sta-
tion point, and the nearest corner of the house about
thirteen feet. It is this relation that obviously de-
termines the *scale* of its perspective representation,
which would be greater if the picture were farther
from the station point, or the object nearer, and *vice
versa*. But we shall come to the question of scale
presently (94).

5

87. The first question is that of the direction to be
given to the various perspective lines; we must de-
The Direc- termine the vanishing points of these various
tions of Hori-
zontal Lines. systems of lines, horizontal and inclined. Ver-
tical lines, and all other lines parallel to the picture, will
of course have their perspectives parallel to themselves
(20). The horizontal lines belong to three systems, the
directions of which are indicated in the plan of the little
house as R and L, going off to the right and to the left,
at right angles to each other; and X, dividing the angle
between them, and making an angle of 45° with each.
If now the spectator, standing at S, looks in a direction
parallel to R, he will see the vanishing point of that sys-
tem of lines directly before his eye; that element of the
system which passes through S is in fact seen endwise,
appearing as a point covering and coinciding with the
vanishing point of the system of right-hand horizontal
lines, which is in the infinitely distant horizon (6). The
perspective of this vanishing point, V^R, in the plane of
the picture, will be found exactly where this element
pierces the picture, that is, where it crosses the line
$p\,p$. V^L and V^X can of course be found in the same
way; and the centre of the picture, C, the point nearest
the station point and at the other extremity of the axis
S C, is easily determined at the same time. Since R and
L are at right angles, the triangle V^R S V^L is a right-
angle triangle, and S lies on the circumference of a semi-
circle of which V^R and V^L give the diameter. In the
picture itself, just above, these points of course appear
on the Horizon; for since these lines are all horizontal,

their vanishing points lie in the trace of the horizontal system of planes (13, I). T R Z and T L Z, the traces of vertical planes parallel to the sides of the building, can now be drawn, as usual, through V^R and V^L; and T P P′, the trace of the diagonal planes, through V^X. The vanishing points of any other horizontal lines and the traces or horizons of any other vertical planes could of course be found in a similar manner. A line perpendicular to the plane of the picture, and accordingly parallel to the Axis, would of course have its vanishing point at C.

88. It only remains to find the vanishing points of the inclined lines M and M′, N and N′, and thence as before, the traces of the roofs, and the vanishing points, P and P′, of their hips and valleys. This is easily done by the aid of the elevations, which show the real inclination of these roofs and gables to be 60° for the lower slope, and 30° for the upper. If the spectator at S, then, while looking at V^R in the direction R, should raise his eyes at an angle of 30°, he would see V^M, the vanishing point of the upper slope of the gable, directly before him, the triangle $V^M V^R S$ being right-angled at V^R. If now this triangle were revolved about the vertical side $V^R V^M$, so as to bring the station point, S, into the plane of the picture at D^R, it would appear in the picture above as the triangle $V^M V^R D^R$, the angle at D^R being 30°.

The Directions of Inclined Lines.

Fig. 13 gives a perspective view of the plane of the picture $p\,p$, with the eye at the station point, S, in front of it; the triangle in question

Fig. 13.

is shown both in its original position and also as it appears when swung round into the plane of the picture.

89. It follows from this that if from V^R the distance $V^R S$ is laid off along the Horizon, we obtain the point D^R; and if from this point we draw a line at an angle of 30°, we shall obtain V^M at its intersection with T R Z. In the same way D^L and V^N may be obtained by setting off the distance of $V^L S$ along the Horizon from V^L, and drawing the line $D^L V^N$, also at an angle of 30°.

The points D^R and D^L are called the right-hand and
Points of Distance. left-hand points of distance; they show the distance of the station point from the right-hand and left-hand vanishing points. It is to be observed that D^R is found on the left and D^L on the right of C.

In the same way the vanishing point of every other horizontal line, as, for example, of X, has its corresponding point of distance, found by setting off along the horizon its distance from S. Thus we have $V^X D^X$ equal to $V^X S$.

90. $V^{M\prime}$ and $V^{N\prime}$ will of course be seen as far below the Horizon as V^M and V^N are above it, and V^P and $V^{P\prime}$ will be at the intersection of the traces of the inclined planes R N and L M, R N′ and L M′, as before.

Fig. 13 gives further illustration of most of these points. To prevent a confusion of lines, the centre, C, is taken on the left-hand side of the picture instead of on the right-hand side, as in Fig. 11.

91. By a reverse process, when the vanishing point of inclined lines is known, their real inclina- To find the true inclination of inclined lines whose perspectives are given. tion can be discovered by drawing a line from this vanishing point to the point of distance of the horizontal line beneath them. Thus in the figure a line from V^P to D^X would give the angle $V^P D^X V^X$, which is the true slope of the line of intersection of the upper roofs.

92. If the lines M and N have different inclinations, the point V^P will of course not come over V^X, and the distance must be measured from the point of the horizon that it does come over.

The vanishing points of the steeper slopes of the lower roofs are found in like manner at V^{M1}, V^{N1}, etc.

93. The exact direction of perspective lines being thus determined, since the position of their The length of lines parallel to the picture. vanishing points is thus exactly fixed, it now only remains to determine their length, and the position of some one point in each. For such lines as are parallel to the picture this is easy, for every such line may be considered to lie in a plane parallel to the picture, the centre of which, or point in the plane nearest the eye, will have its perspective at C, the centre of the picture; the perspective of the line in question will be parallel to the line itself, and its length and its distance from the centre, C, and all other lengths and distances taken in that plane, will be less, as we have just seen (86), in proportion as the distance of the plane from the plane of the picture is greater. If, as in Fig. 11,

the plane $m\,m$ is six times as far from the spectator at

S as the picture $p\,p$, all lines in $m\,m$ will be
drawn at one sixth of their original size, and be
at one sixth their distance from the centre. The front
corner of the house, for instance, which lies in the plane
$m\,m$, is so drawn.

This imaginary plane $m\,m$, which is generally drawn
The Plane of through the nearest part of any object to be
Measures. represented, is called a *plane of measures*,
and, like the picture, is defined in position by the
length and position of its axis, which coincides with
that of the picture, but is generally a great many
times as long.

Its relative distance behind the plane of the picture
is commonly, of course, much greater than that shown
in the figure, in which, for perspicuity's sake, the pic-
ture is represented as being about eleven feet across and
about thirteen feet from the spectator. The picture is
commonly set only a few feet off, while the object rep-
resented is often a hundred times as many.

Sometimes, for convenience, two planes of measures
are employed at different distances, lines in the further
one being drawn to a smaller scale than in the nearer one.

94. It follows from what has been said that any line
Lines in the drawn in a plane of measures in any direc-
Plane of
Measures. tion, horizontal, vertical, or inclined, is also par-
allel to the picture, and that its perspective will be parallel
and proportional to it, but on a smaller scale. This scale
depends on the relative distance of the picture plane
and the plane of measures from the eye, or station point.

If the latter is twice or ten times as far away, lines drawn upon it will be presented in the picture one half or one tenth full size. *All lines in a plane of measures have their perspectives drawn to the same scale.*

It is common, in the case of large objects, such as buildings, to set the picture at just $\frac{1}{96}$, $\frac{1}{144}$, or $\frac{1}{192}$ of the distance of the plane of mea- Scale of the drawing. sures, *i. e.*, of the object. Lines in the plane of measures are then represented $\frac{1}{96}$, $\frac{1}{144}$, or $\frac{1}{192}$ full size, etc. ; that is to say, on a scale of $\frac{1}{8}$, $\frac{1}{12}$, or $\frac{1}{16}$ of an inch to the foot.

The centre of the plane of measures coincides in perspective with the point C, the centre of the picture (93). The perspective of any other point in the plane of measures may be found by laying off its distance, according to the scale, in its real direction, in the plane of the picture.

95. Fig. 12, in which the various vanishing points and traces, and the points of distance D^R and D^L, are determined as in the previous figure, illus- Fig. 12. trates this practice. The picture is supposed to be about six inches from the eye, as in the case of our previous illustrations. This is less than is desirable, but is as much as the scale of these illustrations permits. The front corner of the building, through which the plane of actual measures is taken, is supposed to be a hundred and ninety-two times as far away, that is to say, ninety-six feet from the spectator, or ninety-five feet and a half behind the picture, the scale of the perspective of that corner being one sixteenth of an inch to

a foot. This corner, being eight feet high, is drawn
half an inch high. All other lines in the plane of
measures are drawn to the same scale, which is indeed
the same scale as that to which the plan of the build-
ing is drawn in Fig. 11, and the elevations alongside.
Dimensions can then be transferred directly from these
drawings to Fig. 12, so long as the lines to which
they apply lie in the plane of measures. Such a
line is the line $v\,v$, the line in which the planes of
the front and end walls of the house intersect the plane
of measures. Any such vertical line, lying in the plane
of measures, is called a line of vertical measures. By
Line of Ver- setting off upon this line the heights of any
tical Mea-
sures.   part of the building, by scale, they can be
transferred to their proper places by horizontal lines
directed to the vanishing points. The height of the
gable in the figure is determined in this way. The
line $m\,m$, also, in which the plane of measures intersects
the horizontal plane on which the perspective plan is
taken, is such a line. Its perspective, shown at $g\,l$, is
Line of Hori- called the Ground Line, or line of horizontal
zontal Mea-
sures.   measures, and any dimensions can be laid off
upon it at the same scale as upon the perspective of
the vertical line; as it is parallel to the picture, the
divisions of its perspective are proportional to those of
the line itself.

By prolonging the planes of the other walls until
they intersect the plane of measures, additional lines
of vertical measures are obtained, such as $v'\,v'$, in the
plane of the further gable. In the same way every

horizontal plane gives a line of horizontal measures, as is shown in the case of the two perspective plans below.

96. For very small objects the plane of measures, and with it the object itself, is brought nearer, and may even coincide with the plane of the picture. In this case lines lying in it are drawn full size. *The plane of the picture the plane of measures.*

Sometimes, instead of taking the object of its real size at its real distance, we suppose a miniature of the object to be set up near at hand, of any con- venient scale. In this case also the object may be sup- posed to be close to the plane of the picture, and the plane of the picture to coincide with the plane of measures. *The model or miniature ob- ject.*

This is illustrated in Fig. 11, in which a small plan of the house is drawn in contact with the plane of the picture, pp, just as a large plan, representing full size, is drawn in contact with the plane of measures mm above, the whole being drawn at a scale of sixteen inches to the foot. Or we may regard Fig. 11 as drawn on the same scale as Fig. 12, that is to say, full size, con- sidering the plane mm to be the plane of the picture, six inches from the eye at S, with a miniature of the building, a model made to the scale of a sixteenth of an inch to the foot, just behind it. In this case Fig. 12 represents the picture drawn on the plane mm, just as the little picture in Fig. 11 represents that drawn on plane pp.

97. Having thus the means of drawing in any hori-

zontal or vertical plane a line, lying in a plane of mea-

sures, upon which dimensions can be laid off
by scale, we have now to transfer these dimen-
sions to other lines in the same plane.

If these lines also are parallel to the picture and to
the plane of measures, the case presents no difficulty.
It is only necessary to draw parallel lines from one line
to the other. In the figure, for example, the heights
laid off on the line of vertical measures $v\,v$, at the
front corner are transferred to the other corners and to
other vertical lines by parallel lines directed to the van-
ishing points V^R and V^L. In this way the vertical di-
mensions of every part of an object, and the position of
its horizontal lines, may be determined.

The length of any other lines parallel to the picture,
horizontal, vertical, or inclined, may be obtained in a
similar way from lines in the plane of measures parallel
to them, and lying at the intersection of that plane
with the planes in which they lie.

98. To determine the length of the horizontal lines
not parallel to the picture, and to lay off given
dimensions upon the perspective of such lines,
we can employ a method similar to the Method
of Triangles described in the last chapter. By that meth-
od we laid off upon such lines parts *proportional* to parts
taken upon a line of *proportional* measures. We now
propose to lay off upon such perspective lines parts
equal to parts taken upon a line of *real* measures. Any
triangle will do to transfer proportional parts, but to

transfer equal parts we must have an isosceles triangle; for it is only in isosceles triangles that the parts into which the adjacent sides are divided by lines drawn parallel to the base are equal, each to each.

This is illustrated by Fig. 11, in which the line $m\,m$ at the top, is the line of horizontal mea-
sures. The actual dimensions of the sides Lines of equal measures.
of the house and of the doors and windows
are laid off on this line, and connected with the in-clined lines of the plan by means of lines drawn parallel to the base of an isosceles triangle.

99. It is plain that what is here done in the ortho-graphic plan could be done in a perspective plan if we knew in what direction to draw these parallel lines; that is to say, if we could find the vanishing point of the base of the isosceles triangle.

And this is, in fact, very easy, for a simple inspection of the figure shows that the *point of distance* is the auxiliary vanishing point in question. If the The Point of Distance, again, as a point of equal measures.
spectator at S looks in the direction of the par-
allel lines by which the right-hand line R is
divided, he will see D^R; and in like manner D^L is the vanishing point of the parallels by which distances taken on the line of horizontal measures are transferred to L.

And that this is as it should be is plain from a fur-ther inspection of the figure. For the sides of the isosceles triangles at the top are by construction parallel to the sides of the triangles $SV^R\,D^R$ and $SV^L\,D^L$. These last are accordingly isosceles too, and their two long sides should be equal. The auxiliary vanishing points

then, should be just as far from the vanishing points as
these last are from the station point ; as the points of
distance are, by construction. $V^R D^R$ was originally
taken equal to $V^R S$, and $V^L D^L$ to $V^L S$ (89).

100. The points of distance, then, are the vanishing
points of the parallel lines which will intercept upon
a perspective line parts equal to those intercepted upon
its line of measures.

This is illustrated in Fig. 12, where, in the perspec-
tive plan, parts laid off by scale on the ground line, or
line of horizontal measures, are transferred in their true
dimensions to the perspective lines R and L.　In this
way the length of the walls and the position of the
doors and windows is exactly determined.　These di-
mensions being already shown at the given scale in the
little elevations, it suffices to transfer them directly from
those drawings to the ground line with a measuring strip.

In this way a complete and accurate *perspective plan*
can easily be constructed ; the length of all hori-
zontal lines and the position of all vertical lines
will then be known.　The length of vertical
lines, which gives the position of horizontal ones, is easily
obtained, as we have seen, from vertical lines of measures.

The Method
of the Per-
spective
Plan.

The second perspective plan, above the other, gives
the plan of the roof and dormers.

101. Fig. 11 affords an alternative method of obtain-
ing the horizontal dimensions ; that is to say,
the position of the vertical perspective lines.
If we again regard the plan at the top as the

Fig. 11.

The Method
of Direct
Projection.

plan of a miniature house, or model, set six inches from the eye at S, and regard *m m* as the plane of the picture, in contact with it, we can, by drawing lines from every point in the plan to the station point, find just where every point will appear in the picture; the horizontal dimensions thus obtained can then be transferred directly to the picture in Fig. 12. They are shown by marks on the lower side of *m m*, and will be found to agree exactly with the dimensions obtained from the perspective plan.

This method, which is called that of *direct projection*, is often more convenient than the other, especially when the orthographic plan has previously been prepared, and when, as in the present case, the subject is simple.

But the method of the *perspective plan* is more convenient for designing in perspective, or for Advantages of the Perspective Plan. making a perspective drawing, as often has to be done, from mere sketches. It takes up less room, in the vertical direction; it is less laborious, though requiring perhaps more knowledge and skill; and it has the advantage previously pointed out, — that it enables the position of points at different levels to be separately determined, by the use of separate perspective plans, and enables several successive drawings to be made, if necessary, without repeating the bulk of the labor, since the perspective plan can be made on a separate piece of paper, and used more than once.

Moreover, the points established on a perspective plan explain themselves : it is clear at a glance, and after any lapse of time, which denotes the door, which the

window. In working by direct projection from the orthographic plan, on the contrary, it is almost impossible to remember which point is which, and much time and labor are lost by this confusion.

102. In making a perspective drawing, we generally

The *modus operandi*.

fix first the points V^R and V^L as far apart as is convenient, then strike a semicircle between them, and take the point S in such a position upon it that $S\ V^R$ and $S\ V^L$ shall be parallel to the sides of the object to be drawn. The points of distance and other vanishing points, and the various traces, can then be determined; the perspective plan drawn wherever is most convenient, and the perspective picture erected above it by the aid of vertical lines of measures. It is to be noticed that, since the vanishing points and traces depend solely upon the *direction* of the lines and planes, not upon their *position* (11), the building may be set to the right or to the left of the centre, below or above the Horizon, according as more or less of the left-hand side or of the roof is to be shown.

Care must be taken not to confound the point S with the point assumed o r the front corner of the perspective plan, which is entirely independent of it. The revolved lines, also, $D^R\ V^M$, $D^L\ V^N$, etc., (89) must not be confounded with the traces $V^L\ V^M$, $V^R\ V^N$, etc.; that is to say, with T L M, T R N, etc.

CHAPTER VI.

THE POSITION OF THE PICTURE. — THE OBJECT AT 45°. — THE MEASUREMENT OF OBLIQUELY INCLINED LINES.

IN the last chapter we saw how, when the real length and direction of the lines by which an object is defined are known, the real length and direction of their perspectives may be exactly ascertained. But our investigations covered only the cases of lines parallel to the plane of the picture, whether horizontal, vertical, or inclined, and of horizontal lines inclined to the picture. It remains to find out how to apply a scale to lines inclined to the picture, that are not horizontal, such as the lines of the gables, and of the hips and valleys of the roofs.

Plate V. shows how this is done, and at the same time illustrates some other points of interest. In this plate, as in the previous plate, we have at the side a plan, on a small scale, illustrating the relative position of the spectator at the station point S ; of the object, at A ; of the plane of measures, at $m\,m$; and of the plane of the picture, at $p\,p$. In the present plate the position of the object at A, and of the spectator at S, is the same as in Plate IV. ; but the position of the plane of the picture and of the plane of measures is changed, their previous position being indicated by dotted lines. Their

position is now taken so that the centre at C, and the
vanishing point of 45° at V^X, coincide. That is to say,
the principal lines of the building, R and L, at right
angles with each other, are now at 45° with the plane of
the picture, and consequently with the plane of meas-
ures ; and the line X that divides the angle, making
45° with each, is accordingly at right angles with the
picture and the plane of measures, and is parallel to the
Axis. V^X, then, of course, comes in the same place that
C does.

103. Now, in the first place, this illustrates the point
that, the position of the spectator and of the object being
given, it is purely a matter of convenience how we take
Position of   the plane of the picture, and in what direction
the plane of
the picture.   its axis is drawn. In Plate IV., Fig. 11 and
Fig. 12, the axis was directed towards the end of the house,
C coming near the corner, and the left-hand side of the
house was less inclined to the plane of the picture than the
Fig. 14.   right-hand side. In Plate V., Fig. 14 and Fig.
Fig. 15.   15, the axis is directed further to the left, C co-
inciding in position with V^X, and the plane of the picture
and plane of measures are equally inclined to both sides
of the house, making 45° with each. But of course this
would not change the appearance of the object A, as
seen from S ; and if the drawings were both made so as
exactly to cover and coincide with it, the two represen-
tations would look exactly alike when viewed each from
its own station-point, so far as concerns their main out-
lines. These are not affected by the alterations and

repairs the house has undergone since we last saw
it.

104. In the second place, the plate illustrates the
point that it is a much simpler thing to make The object
and the pic-
a drawing in perspective when the plane of ture at 45°.
the picture, as in Fig. 15, is at an angle of 45° with
both sides of the object than when it lies accidentally,
as in Fig. 11.

For this attitude of the object sets the station-point
at its maximum distance from the picture, The symme-
try of the
which is obviously an advantage. This dis- vanishing
points and
tance, S C, is in this case just half the length horizons.
of the Horizon between V^R and V^L. Moreover, since S
is equidistant from these points, and C, which is also
V^X, is half way between them, it follows that D^X coin-
cides with V^R, and also that there is another D^X coincid-
ing with D^L; that D^L and D^R are also equidistant from C ;
and, in short, that the whole series of vanishing points
and points of distance is symmetrical about T P P' in
one direction, as it is about the Horizon in the other.

105. From this symmetry, which is clearly exhibited
on a small scale in Fig. 14, it follows that V^M and V^N
are equally distant from the Horizon, that T L M is
parallel to T R N', and T R N to T L M', and hence that
V^Q and $V^{Q'}$, which lie at the intersection of these traces,
are at an infinite distance. Hence the lines Q and Q' are
both parallel to the picture, the perspective of Q is par-
allel to T R N and T L M', that of Q' to T L M and
T R N', and the projections of these lines in the per-
spective plan are parallel to the Horizon.

6

106. The great convenience of this is shown in Fig.

15, where the hips of the roof of the school-

The practical
conveniences house on the left, that form its outline against
of this.

the sky, are drawn parallel to the traces of
the planes in which they lie instead of being directed,
as in Fig. 13, to their inaccessible point of intersection.

Accordingly objects are generally drawn at 45°, being
set to the right or to the left of the centre, as it is desired
to expose their left side or their right side more fully to
view (102).

107. It will be observed that the relation of the prin-
cipal distance points, D^R and D^L, to their corresponding
vanishing points, V^R and V^L, is the same as that of
the corners of an octagon to the corners of the square
from which it is cut, the distance of each from the

remoter corner being half the diagonal of
Fig. 16.
the square. Compare Fig. 16 with the small
plan in Fig. 14. The distance apart of the vanishing
points then being as 200, the distance of the station-
point from the picture at C will be as 100, and each
point of distance will be at a distance of 141 from its
own vanishing point, and 59 from the other. Or, in
other words, the distance apart of the vanishing points
being as 10, that of the points of distance from the
centre, C, will be as 2, their distance from the nearest
vanishing points will be as 3, their distance from each
other will be as 4, and the distance of the spectator
from the picture as 5, very nearly.

108. Fig. 17 shows the amount of the error involved
Fig. 17.   in this assumption. It is so slight that for

practical purposes the distance from the centre to the points of distance may generally be taken as two fifths of that to the vanishing points, when the picture is at angles of 45° with the sides of the object.

109. Fig. 15 also illustrates another point of capital importance, showing how dimensions are laid Objects nearer than the plane of measures. off by scale upon lines lying on the *hither* side of the plane of measures; these lines are necessarily drawn to a larger scale than that used for lines in that plane, just as lines beyond it are drawn to a smaller scale. The principle of the isosceles triangle, explained in the last chapter, and again illustrated in Fig. 14, still holds good.   The fence, for instance, that is built out from the front corner of the house, toward the right, has the position of its posts laid off upon the line of horizontal measures, or ground line, $g\,l$, to the right, by scale, and then transferred to the line of the fence in the perspective plane, by parallel lines drawn to D^L, the fence being in a line with the left-hand side of the house.   These lines diverge as they come forward, and the scale of the fence is obviously increased.   The building on the left is entirely in front of the plane of measures.

110. To avoid confusion the points on the ground line intended for use beyond the plane of measures are indicated above it, and those which give the dimensions of objects this side of it, as is the case with this fence, are indicated below it.   A better way in some cases is to have separate ground lines for figures in front of the plane of measures.

111. Let us now take up the question left unanswered

in the previous chapter, and see how dimensions can be laid off by scale upon obliquely inclined lines; upon lines, that is to say, which, like the gables and the hips and valleys of roofs, lie in inclined planes, and are hence inclined not only to the plane of the picture but also to the horizontal plane.

In the first place, it is clear that just as every horizontal plane has a line of horizontal measures, or ground line, where it intersects the plane of measures, and just as every vertical plane has a line of vertical measures which is its line of intersection with the plane of measures, so every inclined plane, such, for example, as a roof, in like manner has its own line of measures lying in it and also in the plane of measures, upon which dimensions can be laid off by scale. It is the line in which that plane intersects the plane of measures, and it passes through the point at which any line in that plane pierces the plane of measures. This line of measures and its perspective are always parallel to the trace of the given inclined plane. For, as we have seen (78), this trace is the intersection of the plane of the picture with a plane parallel to the given plane, and passing through the eye, while the line of measures is the intersection of the given plane, with a plane parallel to the picture. The intersections of these two sets of parallel planes must be parallel.

112. To find a line of real measures in any plane, then, it suffices to find the point at which any line in it

pierces the plane of measures, and to draw through this point, when found, a line parallel to the trace of the plane.

113. In Fig. 15, for example, we have several such lines : $g\,l$ parallel to the Horizon, $l\,m$ parallel to T L M, $n\,r$ parallel to T N R, etc. In the

Fig. 15.

case of the ground line, the horizontal lines to be measured off are prolonged until they reach the plane of measures, and the line $g\,l$ is drawn parallel to the horizon through the point thus obtained.   In the case of the inclined lines of the gables, that on the end of the building at the right is brought forward till it touches the plane of measures, and the corresponding line on the porch roof of the porch of the little school-house on the left is carried back until it reaches the plane of measures.   These points are ascertained from the perspective plan, the point where the horizontal projection of these inclined lines pierces the plane of measures being the projection of the point where the lines themselves do so.   The lines of measures, $l\,m$, are then drawn through the points thus ascertained parallel to the trace of the plane L M.   As these gable lines lie also in the vertical planes, R Z, vertical lines of measures, $r\,z$, parallel to T R Z, may also be drawn through the same points.

114. In the case of the hip line, Q, of the little school-house, which is parallel to the plane of measures and to the trace T R N (105), it is necessary to proceed as in the case of vertical lines, or any others that are parallel to the picture, and carry across it a line lying also in the plane R N, but inclined to the picture, by means of which the line where the plane intersects the

plane of measures can be found.   In the figure the line
of the other hip, P, is used for this purpose, its horizon-
tal projection being found in the perspective plan.   It
would have done just as well to prolong the line of the
eaves until it met the plane of measures, and to draw the
line *r n* through the point thus found.

115. All these lines of measures lie in the plane of
measures, and are consequently drawn to the same scale.
On all of them one sixteenth of an inch is the true
perspective representation of one foot, and any dimen-
sions laid off on this scale can be transferred to any other
lines in the inclined plane, such as the lines of the
gables, by means of points of distance lying in the trace
or horizon of the inclined plane, just as dimensions can
be transferred from the ground line to lines lying in the
horizontal plane by means of points of distance taken
on the Horizon.

116. The points of distance upon these inclined
Manifold   traces, or horizons, are easily obtained from
points of dis-
tance.   those upon the horizontal Horizon, if we may
use such an expression.   It is obvious, indeed, that the
problem is a simple one, if we remember that the point
of distance corresponding to any vanishing point is
found by setting off upon the horizon passing through
that point the true distance of that point from the sta-
tion-point.   If this is done, the triangle S V D formed
by the three points will, of course, always be isosceles
by construction.   Now, the distance of every vanishing
point from the eye is known, and can readily be laid off
on all the traces, or horizons, that pass through it.

117. The distance of V^R from S, for example, is known to equal $V^R D^R$, and this distance laid off on T R N, T R N′, T R Z, and the traces of any other planes that contain R, give the corresponding points of distance on those traces, D^{R1}, D^{R2}, D^{R3}, D^{R4}, etc., all, in fact, lying at equal distances about V^R on the circumference of a circle whose radius equals V^R S, the distance of the vanishing point in question from the eye. This is shown both in Fig. 14 and in Fig. 15. In like manner a circle may be drawn about V^L, containing the distance points that belong to that vanishing point. Its radius will be $V^L D^L$, the distance of V^L from the eye.

The *locus* of points of distance.

Fig. 14.

Fig. 15.

Such a circle is the *locus* of the points of distance.

118. The distance of V^M from the eye is $V^M D^R$, as is obvious if we suppose the triangle $V^M D^R V^R$ revolved back around its side T R Z until D^R is at S again. A circle drawn around V^M, with $V^M D^R = V^M$ S as a radius, will accordingly give points of distance, by which any inclined line M, vanishing at V^M, may be measured off by lines of measures lying in any of the planes that contain it, and whose traces accordingly pass through V^M.

As we have taken the lines M and M′ at an angle of 30° with the Horizon, the angle M D^R M′ is 60°, the triangle M D^R M′ is equilateral, and D^M falls upon $V^{M′}$, and $D^{M′}$ upon V^M.

In every case, of course, the line of real measures, like the lines of proportional measures discussed in a previous chapter, is drawn parallel to the trace of the plane containing it (76).

119. It will be noticed that there are two points of Two points of distance on T R Z, at equal distances above distance on each horizon. and below V^R, just as we found that both V^R and V^L would serve as points of distance to V^X. Either of these can be used instead of D^R to set off given lengths on the right-hand horizontal lines. In the upper perspective plan, for example, the length of the right-hand side of the house is set off not only on the ground-line, and transferred by means of D^R, but also on the vertical line of measures, both above and below the ground, and transferred by means of D^{R2} and D^{R3}, giving in every case the same points.

Indeed, it is practicable to have a second point of distance on all the traces, beyond V^R. They Fig 18. would be the vanishing points of the bases of *obtuse-angled* isosceles triangles. Such a triangle is shown in Fig. 18, *b*. Its employment is shown in the upper perspective plan, where the length of the right-hand side of the house is laid off by scale on *g l*, to the *left* of the corner, and transferred to the perspective line by means of the point D^{R4}, which is out of the picture.

120. If, then, through the point at which any perspective line pierces the plane of measures, a line of measures be drawn parallel to the trace of any plane that contains the line (that is to say, parallel to any trace passing through the vanishing point of the line), any distance laid off by scale on the line of measures may be transferred to the perspective line by means of a point of distance in that trace ; and this point of distance will be the point where that trace is cut by the

circle, which is the *locus* of the points of distance, — a circle whose radius equals the distance of the vanishing point of the perspective line from the eye, and whose centre is at the vanishing point.

This is illustrated in the lower perspective plan, in which a certain distance, being that from the corner of the house to the further edge of the window, is laid off on several lines of measures, $g\,l$, $l\,z$, and $l\,m'$, parallel to the several traces that meet at V^L. Lines drawn to the points D^L on those traces all give the same point upon the line L.

121. As the points taken upon the line of measures are all equidistant from the corner of the plan, they lie in the circumference of a small circle, which is the *locus* of the points of measures, just as the corresponding points of distance lie in the circumference of a large circle; and we have the curious phenomenon of lines drawn from certain points on the small circle to corresponding points on the large one, all intersecting at the same point, upon a line joining the two centres. But Fig. 19, which presents these geometrical relations in a diagram, shows that there is nothing surprising in this.

The locus of points of equal measures.

Fig. 19.

122. Now, just as any line drawn in any direction, at random, through one end of a perspective line, may be used as a line of proportional measures (81), so any such line that lies in the plane of measures may be taken as a line of real measures; and any dimensions taken upon it by scale may be transferred to any perspective line that touches it,

Random lines of equal measures.

by means of a trace drawn through the vanishing point
of the perspective line, parallel to the random line; the
point of distance being taken upon that trace at the
same distance from that vanishing point as its other
points of distance.

This also is illustrated in the lower perspective plan,
where, in addition to the lines of measures drawn par-
allel to the traces already obtained, a line of measures,
$l\,t$, is drawn at random at an angle of 15°. A ˙new
trace, T L T, is drawn through V^L parallel to it, and a
new point of distance, D^{L5}, obtained upon it, where it
intersects the *locus* of D^L.

The same thing is done again in the picture Fig. 15,
Fig. 15.   where the divisions of the fence, on the right
of the house, are taken by scale on an arbitrary line of
measures just above. As this also is taken at 15°, the
same point of distance, D^{L5}, serves to transfer the dimen-
sions to the line L, the top line of the fence.

123. Finally, just as of any two perspective lines
having the same vanishing point, one may be
Perspective
lines as aux-   regarded as the trace of a plane passing through
iliary hori-
zons.   the other (82), so a point of distance taken
upon the first, at the proper distance from that vanish-
ing point, may be used to lay off given dimensions upon
the second, by using a scale upon a line of real measures
drawn parallel to the first, from the point where the
second pierces the plane of measures.

This is illustrated in the plan of the little school-
house shown in the plate. The front line of the plan is

regarded as the trace of a plane, and upon this trace the point of distance D^{R5}, upon the *locus* of D^R, is easily determined. A line of measures is drawn parallel to it through the further corner of the plan, where the line R on the back of the building touches the plane of measures. The true width of the building being laid off on this line of measures by scale, and transferred to the perspective of the further side by means of a line drawn to the point of distance, we obtain the left-hand corner of the building, and the position of the rear window, as we ought to.

124. Here, as before (84), we must be careful not to suppose that this line of measures is drawn on the floor of the school-house. It lies, in fact, in the vertical plane of measures, being an inclined line parallel to the plane of the picture. So, also, the point of distance is not on the ground, but is in the infinitely distant horizon which the front line of the building covers and apparently coincides with.

125. It is to be observed that the methods of measuring off perspective lines described in this and the preceding chapters are strictly analogous to the method of dividing perspective lines described in the previous chapter. By that method, called the Method of Triangles, the perspective of a finite line is divided up, in any desired proportion, by drawing from one end of it a line, called a line of proportional measures, parallel to the plane of the picture, and then setting off upon this line, at any

Proportional measures, equal measures, and scale measures.

convenient scale, dimensions proportional to those de-
sired.   These are then transferred to the perspective of
the given line by means of a point of proportional mea-
sures found upon the trace or horizon of the plane in
which both lines lie.   This horizon passes through the
vanishing point of the given line, and the point of pro-
portional measures is the vanishing point of the base of
the scalene triangle, the other two sides of which are
the given line and the line of measures.

By using the point of distance, however, as a point of
measures, the triangle becomes isosceles, and the parts cut
off upon the perspective line are not only proportional
but equal to those taken upon the line of measures.   It
is now the line of measures that is of fixed dimensions,
and the perspective line is not divided up, but has these
dimensions set off upon it, being treated as a line of
indefinite length.   In the Method of Triangles the line
to be divided is of definite length, the line of propor-
tional measures is an indefinite line, and the point
of (proportional) measures may fall anywhere in the
horizon of the plane of the triangle.   In measuring a
given distance upon a perspective line, on the other
hand, the line of equal measures, or of scale measures
if a scale is used, is a definite portion of the ground line,
or other line of measures, the given line is of indefinite
length, and the point of (equal) measures is the point
of distance.

In dividing a given line proportionally, the length of
the line is indeed always fixed, and that of the auxiliary is
indefinite ; in transferring real dimensions, on the other

hand, these fix the length of the auxiliary, and it is upon an indefinite length of given line that they are to be set off. This difference is illustrated in Fig. 18, Plate V. In *a* the given line is of definite length, the auxiliary of indefinite ; in *b* the contrary is the case.

Moreover, it is to be observed that although a line of equal measures, like a line of proportional measures, may be drawn parallel to the picture from any point in the perspective line, it will not serve for scale measurement unless it lies in a plane of measures whose position has been already determined. It is accordingly necessary to extend the given line until it touches or pierces the established plane of measures, before a line of scale measures can be drawn, as has frequently been exemplified in this chapter. The difference is illustrated by comparing the treatment of the gable in Plate III. with that of the gable in Plate V. In the latter the line of the gable is extended so as to bring the lines of measures into the same plane as all the other lines of measures.

CHAPTER VII.

PARALLEL PERSPECTIVE. — CHANGE OF SCALE.

THE last chapter discussed the case in which the plane of the picture, and consequently the plane of measures parallel to it, is set at *an angle of 45°* with the sides of the object, so that the "vanishing point of 45°," V^X, coincides with C, the centre of the picture, and the principal vanishing points of the right-hand and left-hand lines, V^R and V^L, and their points of distance, D^R and D^L, are symmetrically disposed on each side of it. The other diagonal, at right angles to X, is in this case, of course, parallel to the picture, and to the Horizon.

126. Let us now consider the analogous case in which the plane of the picture, and consequently the plane of measures, is taken parallel to one of the principal sets of lines, and at right angles to the other. This is illustrated in Plate VI. When objects are thus represented with one side parallel to the picture, and the adjacent sides perpendicular to it, they are said to be drawn in Parallel Perspective.

Plate VI.

127. The relation between this case and that discussed in the last paper is shown in the two buildings upon the quay, on the left-hand side

Fig. 20.

of Fig. 20. The nearest one, whose roof rises just above the rail of the descending steps, stands at 45° with the picture, just as in Plate V., having the vanishing points of its main lines at V^R and V^L; while one set of its diagonals, X, as seen in the perspective plan below, converges at C, $= V^X$, and the other is parallel to the Horizon ; the lines of the hips, P and P′, being directed to V^P and $V^{P\prime}$, and the hips Q and Q′ being parallel to the traces T R N and T L M (105). The points V^M and V^N are off the picture, but V^P suffices to determine the position of these traces. These traces are not shown in the picture, to avoid confusion.

128. The larger building beyond, like all the other objects in the picture, is drawn in Parallel Parallel Perspective. It is set at an angle of 45° spective. with the building just mentioned, the sides of one being parallel to the diagonal lines that divide the angles of the other, and *vice versa*. This is rendered more obvious by comparing their perspective plans.

129. These plans, it will be noticed, are drawn wherever it is most convenient to put them, that of the further building being taken so very far below the ground as to come lower down on the paper than that of the nearer building.

130. Now, it is to be observed that while the planes of the nearer roofs, as in the previous plates, Inclined have for their traces the lines T R N, T L M, planes, with one element etc., and the hips have their vanishing points parallel or normal to at V^P, $V^{P\prime}$, etc., the roof of the further build- the picture. ing, and the tops of the posts in the foreground, which,

for convenience, are given the same slope, present a new case, which we have not hitherto met, — the case of inclined planes whose horizontal element is either parallel or perpendicular to the picture.

131. The several flights of steps, also, ascend and descend along inclined planes, either at right angles to the picture, or, as is the case of those at either end of the platform in the extreme foreground, parallel with it. Let us take these flights of steps first, and make the rise of each step six inches, and the tread nineteen inches. Now, as the flights in the foreground are parallel to the picture, they will be drawn in their true proportions, and will give the true slope of the steps; and if we suppose the plane of measures to coincide with the front of the little pavilion, or with the further end of the steps, we can lay off the steps by scale at once, and ascertain their true slope. It proves to be about $20°$.

132. To ascertain the point V_1, which determines the direction of the sloping lines of the flights which ascend at right angles to the picture, we have only to draw a line at V^L, $= D^X$, making an angle of $20°$ with the Horizon. Its point of intersection with $T P P'$, above V^X, will be the point in question (88). V_1', the vanishing point of the descending flights, will be at an equal distance below V^X.

133. The traces of the inclined planes in which these steps lie, pass, of course, through the vanishing points of all the lines that lie in them (13. II.). V_1 and V_1' are the vanishing points of the steepest lines of the ascending and descending planes, their horizontal elements being par-

allel to the picture, and having their vanishing points at an infinite distance to the right and left. The traces drawn through V_1 and V_1' are accordingly drawn horizontal, or parallel to the Horizon. The trace of a plane is, indeed, always parallel to that element of the plane which is parallel to the plane of the picture (76); for that element has its vanishing point on the trace, but at an infinite distance. For distinction's sake we have lettered these two horizons H_1 and H_1', just as the Horizon is lettered H.

134. The trace of the inclined planes of the flights of steps that ascend to the right and go down to the left, on the edges of the picture, passes of course through the point C, the vanishing point of their horizontal element, and is parallel to the steepest line of the slope, which is the element parallel to the picture. This trace is not shown.

135. The traces of the inclined planes of the roof of the building on the left, and of the little flat pyramids on top of the posts, are determined in the same way. The slope of these planes is about 25°; they are accordingly steeper than the slope of the steps; and the vanishing points V_2 and V_2', found by drawing lines from V^L or V^R at an angle of 25°, are further from the Horizon than V_1 and V_1'. H_2 and H_2' are the traces of the front and back planes, while the planes of the right-hand and left-hand sides pass diagonally across the picture through C at an angle of 25°. They are lettered $T M_2 N_2'$ and $T N_2 M_2'$. The vanishing points of the hips, or angles of the pyramids, are at the intersections of these traces

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with H_2 and H_2', at the points marked V^{M2}, V^{N2}, $V^{M2\prime}$, $V^{N2\prime}$. They are, of course, in the traces $T\,L\,Z$ and $T\,R\,Z$, since these hips and angles obviously lie in vertical planes making 45° with the picture.

136.　The only other object in the figure, the pavilion or belvedere, a little building just twice as long **Dimensions.** within as it is wide, is easily drawn. All the lines of the perspective plan are either parallel to the Horizon or converge to C, which is V^X, and are cut off at the length required by setting off the true length by scale, upon $g\,l$, and transferring it to the perspective line by a line drawn to the point of distance, V^R, $= D^X$. The intermediate points are determined in the same way. The front of the building being in the plane of measures, all its parts are drawn to scale, proportionally to their real dimensions, and the same is true of the further end of the building and of the arched wall in the middle, only that being more distant, the scale on which they are drawn is smaller. All the arches are struck from the perspective of their centres, which, being in reality all on a horizontal line perpendicular to the picture, occur in the perspective on a line drawn from the centre of the front arch to C. Their exact position on that line is determined on the perspective plan.

In the same way the rafters in the roof may be laid off exactly, two feet apart, using the point of distance $D^X = V^R$, or, if their number is known, the space they occupy may be divided into six equal parts by using the method of triangles, described in the fourth chapter.

137. For the length indicated in the figure, and with the station point so near the picture, there is Change of no practical inconvenience in thus using V^R or Scale. V^L as points of distance. But if, as might easily be, the room to be drawn were twice as long as this, the point g would be inconveniently far away; and if the station point were farther from the picture, — and it is always an object to have the station point as far away as the point from which the picture will generally be regarded, — say two feet, the point V^R also would be practically inaccessible.

138. These inconveniences, which are likely to occur in oblique perspective as well as in parallel perspective, may in all cases be got over by substituting another triangle for the isosceles triangle hitherto employed. Instead of using a triangle whose legs are equal we may just as well employ a scalene triangle, provided only the ratio of the two legs is known. Lines drawn parallel to the base will not now indeed divide the adjacent sides into equal parts, but we can just as easily as before cut off any required dimensions.

139. This is illustrated in Fig 21, c. We have here, as in Fig. 12 and Fig. 14, the plane of the picture Fig. 21. $p\,p$ in immediate contact with the object, which Fig. 12. is here the model of a small room or passage, Fig. 14. whose plan, as above, occupies two squares. Let us suppose the spectator to be at S_1, at a distance from the picture of $S_1\,C$, the length of the Axis, X. V^R, then, or D^X, will be at an equal distance to the right, as shown, on the prolongation of $p\,p$. This gives the

right-angled isosceles triangle, $S_1 C D^x$, and the length
of the room laid off on $p\,p$ in the other direction may be
transferred to the side of the room by the line $D\,d$,
drawn parallel to $S_1 D^x$.   Fig. 21, b, shows this done in
perspective, determining the point d, as just

now in Fig. 20.

140. But this point d will be fixed with equal pre-
cision if we take instead of D^x another point, $D_{\frac{1}{2}}$, half
way from C to D^x and make the base of our triangle
$S_1 D_{\frac{1}{2}}$.   The triangle is no longer isosceles, but we know
that lines drawn parallel to the base will make the seg-
ments of the short side just half as long as those of the
long side, and that does just as well.   For if we now
lay off in the other direction, just as before, half the
length desired, a line drawn parallel to $S_1 D_{\frac{1}{2}}$ will give
d, just as before.   And in like manner we might take
$D_{\frac{1}{4}}$, half way between C and $D_{\frac{1}{2}}$, and lay off one quarter
the required length of the room, etc., with the same
result.

141. Applying this now to Fig. 20, by taking $D_{\frac{1}{2}}$ or

$D_{\frac{1}{4}}$, measured off along the Horizon from V_x at
half or quarter the distance of the station point
from that vanishing point, we can cut off any desired di-
mensions on the perspective lines that converge at V^x by
laying them off upon the ground line, $g\,l$, by a scale of
equal parts half or quarter as large as those used in the
plane of measures, an l employed for the horizontal and
vertical lines in that plane.   Instead of using a half-
inch scale, and using D^x as the point of distance, we
may use a quarter-inch scale, and transfer the dimen-

sions to X by means of $D_\frac{1}{2}$, or an eighth scale and use $D_\frac{1}{4}$.

142. These results may be summed up as follows : —

Auxiliary points of distance, which may be called points of half distance, quarter distance, etc., Points of halfdistance, etc. may be obtained by laying off from the vanishing point of any line, upon any horizon or trace passing through that vanishing point; a half, or a quarter, etc., of the real distance of the station point from that vanishing point ; the required dimension must then be laid off in an opposite direction upon a line of half or quarter measures, etc., drawn parallel to that trace through the point where the perspective line in question touches the plane of measures, the scale employed being a scale of equal parts, half, or quarter, etc., as large as those employed for lines in the plane of measures.

143. It is sometimes convenient also to use a smaller scale for lines parallel to the plane of measures and at some distance behind it, whether vertical, horizontal, or inclined. This is tantamount to establishing another plane of measures, and another ground line, two, three, or four times as far off, and diminishing the scale used in the picture accordingly.

144. This use of points of half distance, third distance, or quarter distance, and this employment of an auxiliary plane of measures, and the change of scale involved in both these devices, are obviously just as practicable in other cases as they are in this. But they are most often used in the case of parallel perspective. For in oblique perspective the need of having the vanishing

points within convenient distance generally limits the
distance of the station point and keeps the points of dis-
tance near at hand.   Both are always nearer than the
remoter vanishing point.   But in parallel perspective one
of the principal vanishing points is at an infinite dis-
tance, and the points of distance, though nearer than
that, may yet be quite out of reach.   The use of points
of half and quarter distance, etc., enables one to set the
station point as far away as he pleases.   There is
absolutely nothing to prevent his taking the point of
view most favorable for his purpose, the point of
view, namely, that will give the best proportions to his
picture.

145. In practice it is most convenient *to assume the*
An inverse   *desired proportions* at the outset ; that is to
procedure
common.   say, having determined on the scale at which
the nearest end of the street or room is to be drawn,
to make the further end as large, and to set it as far on
one side and as far up or down, as will look best.   Van-
ishing· lines drawn through the corresponding points of
the two ends will then determine the centre, C, and the
Horizon.   This is all that need be determined, since the
length of the room or street is supposed to be known.
If from the near end of the perspective of this length,
the real length is laid off upon the ground line, at any
convenient scale, and the last point connected with the
further end of the perspective line, and prolonged until
it meets the Horizon, the point thus ascertained will be
a point of half, quarter, or third distance, according as

the scale chosen is a half, a quarter, or a third of that used for the plane of measures. The corresponding distance of the station point in front of the centre, C, will then be two, three, or four times the distance of the centre from this auxiliary distance point.

The two views of the street in Fig. 22, both of which were drawn in this way, illustrate the impor-   Fig. 22. tance of carefully proportioning the parts of the picture. The upper one shows the street as it would appear quite near at hand, much reduced ; the other, drawn to the same scale, shows how it would look at a greater distance.

146. Inasmuch as the relative size to be given to the two ends of a room, drawn in parallel perspective, or to the two ends of a street, depends thus entirely on the position of the spectator, and not at all on the real length of the side, it follows that a long room seen from one point may be drawn to look just like a short room seen from a nearer point, and that there is no knowing which is which. This is illustrated in Fig. 21, where the front half of the room in the upper figure,   Fig. 21. a, as seen from S_2 is just the shape of the whole room in the lower figure, b, as seen from S_1. The same drawing also, if regarded from different distances, may give the impression of a long street or room, or of a short one.

147. In sketching interiors on the spot, the point of view is generally excessively near. Such sketches, being viewed from a greater distance, generally give too great an impression of length, and often have to be re-

drawn, so as to show the room as it would look from a point which really is outside of it.

148. Parallel perspective is not often used for a single Practical object, inasmuch as in order to show a sec-
Limitations
in the use of ond side, at right angles to the picture, it is
Parallel Per-
spective. necessary to set it a good way from the centre.
Fig. 23 shows the top of a chimney, the end Fig. 23.
of which is parallel to the picture, while the long side is perpendicular to it. The horizontal lines of the brick-work on this side are directed to C, those on the end are parallel to the Horizon, while the vanishing lines of 45°, as seen on the perspective plan below, are directed to V^L and V^R.

149. It is to be noticed that all the surfaces on the end of the chimney are drawn of their true shape and proportion, as if seen in elevation. Still the whole end of the chimney is not drawn in elevation, the relations of the several parts being changed and the symmetry of the whole disturbed, since the nearer surfaces are set fur-ther to the right and higher up, and since something is seen of the horizontal surfaces that separate them, which in the elevation just below, drawn in orthographic pro-jection, are not seen at all. The chimney certainly looks very ill drawn, and it is not easy, even by keeping the eye sedulously at the station point, opposite C, to make it look quite right.

150. The use of parallel perspective is accordingly pretty much confined to cases where two objects are to be shown, one on the right and one on the left, as in

street views, or interiors. In these cases the eye natu-
rally takes a central position, opposite the middle of the
street or the middle of the room represented. It is not
necessary, of course, that the Axis should be exactly
in the middle, and it is generally taken near one side,
so as to show as much as possible of the other, and thus
prevent an absolute symmetry.

The use of points of half and quarter distance is
further illustrated in Chapter XV., section 333, which
describes the method of drawing lines at 45°, directed
towards D-V^{x}, when only D$\frac{1}{2}$ or D$\frac{1}{4}$ is accessible.

CHAPTER VIII.

OBLIQUE, OR THREE-POINT PERSPECTIVE.

151. THE last chapter discussed the phenomena of
Parallel Perspective, in which, of the three
sets of lines that define a rectangular object,
two are parallel to the picture and have their
vanishing points accordingly at an infinite distance; the
third alone has its vanishing point in the plane of
the picture. This may be called, accordingly, "One-
Point Perspective," since it employs only one vanish-
ing point, V^X at C.

"One-Point" or Parallel Perspective.

152. In the previous chapters only one of the princi-
pal sets of lines, namely, the vertical lines,
were parallel to the picture, both sets of hori-
zontal lines being inclined to it at an angle,
one to the right and one to the left. This may accord-
ingly be called "Angular" or "Two-Point Perspective,"
two vanishing points being employed, V^R and V^L.

"Two-Point" or Angular Perspective.

153. We now come to a third case, that in which all
three of the principal lines of a rectangular
object are inclined to the picture, the object
presenting towards the eye a solid corner.
In this case, all three vanishing points are employed,
and the drawing may be said to be made in "Ob-

"Three-Point" or Oblique Perspective.

lique," or " Three-Point Perspective."   Plate VII. illus-
trates this case, Figs. 24, 25, and 26 present- Plate VII.
ing examples in which, though the object is Figs. 24, 25, 26.
vertical, the plane of the picture is inclined ; while in
Fig. 27 the picture is vertical, as usual, but   Fig. 27.
the cubical block on the floor, the two covers of the box
in the foreground, and the chair, are all tipped so that all
their edges are inclined to the picture.   They are accord-
ingly drawn in Three-Point Perspective.

154. Fig. 24 shows a post at the corner of a fence as
it appears when one looks down upon it, the   Fig. 24.
plane of the picture being inclined backwards at the top.
Fig. 25 is a drawing of the tower of old Trin-   Fig. 25.
ity Church, in Boston, which was destroyed by the fire
in November, 1872, as it appeared when one was looking
up at it, the top of the picture being inclined forward.
Fig. 26 is a similar view of the tower and spire   Fig. 26.
of Salisbury Cathedral, taken from a photograph.   The
vanishing points in Fig. 24 are at V_1, V_2, and V_3; those
of Fig. 25 at V_4, V_5, and V_6; and those of Fig. 26 are
not shown, but can easily be found.

155. To make these drawings look *natural*, the paper
should be held at an angle, below the eye for the first
and above the eye for the other two.   The vanishing
points of the vertical lines, V_3 and V_6, should be just
above or below the eye.

156. Fig. 27 illustrates all three kinds of perspective,
the room being drawn in parallel perspective,   Fig. 27.
with only one vanishing point, at C; the bookcase and
the box in angular perspective, with two vanishing

points, at V^R and V^L; and the lids of the box, with the chair and the cubical block, in Three-Point Perspective, with vanishing points at V^L, V^M, and V^O. The three sets of planes, as marked on the cube, are of course L M, L O, and M O; and their traces, T L M, T L O, and T M O, form a triangle lying between the three vanishing points.

157. A plane of measures is supposed to pass through the nearest corner of the cube. In this lie three lines of measures, $l\,m$, $l\,o$, and $m\,o$, parallel to the three traces (79). On each of these lines the real length of the edge of the cube is measured off, giving the six points l, l, m, m, o, o. These dimensions are transferred to the three edges of the cube, L, M, and O, by means of the points of distance D^L, D^M, and D^O, which indicate the distance of each of these vanishing points from the station point, S, in front of the picture, opposite C. Each of these points of distance occurs twice, once on each of the horizons that meet at its vanishing point (120).

158. In the same way the width of the box is laid off on a vertical line passing through its front corner, and transferred to its right-hand edge by means of D^R, in the trace T M O or T R Z. Half the width of the box, which is the width of each half of the cover, is transferred to the inclined lines of the cover, directed to V^M and to V^O, by means of the points of distance, D^M and D^O, on T M O.

159. This is all exactly in accordance with what has

been done in previous cases, and involves no new prin-
ciple. The only new question which oblique
perspective presents relates to the position of
the station point, S, of the centre, C, of the
various points of distance, and of the three
vanishing points. Their relations are ob-

viously much more strictly defined than in the previous
cases. For, in One-Point Perspective, the vanishing
point, C, being given, the station point may be any-
where upon the Axis, the line passing through C in a
direction perpendicular to the plane of the picture.
In Two-Point Perspective, the vanishing points V^R and
V^L being given, the station point, S, must be somewhere
on the circumference of a semicircle, whose diameter lies
between those points, and which is itself in a horizon-
tal plane perpendicular to the plane of the picture. But
any point in this semicircle will do. In Three-Point
Perspective, it must in like manner lie somewhere in
the circumference of each of three semicircles, whose
diameters are the three sides of the triangle formed by
the three vanishing points. For, since the three edges
of the rectangular object form right angles with each
other, the lines drawn from the eye to the vanishing
points parallel to those edges must also be at right
angles with each other. These three lines, in fact, to-
gether with the three traces, form a triangular pyra-
mid, the vertex of which, at S, is composed of three
right angles. This pyramid is of the same shape, obvi-
ously, as the small triangular pyramid that would be
formed by cutting across the corner of the object rep-

resented with a plane parallel to the plane of the pic-
ture. The lines of intersection in each plane would of
course be parallel to the trace of that plane (78). The
Fig. 27.   corner of the cube in Fig. 27 is represented as
cut across in this way.

160. Now it is obvious that only one such pyramid
can be constructed upon a given triangle as a base; that
is to say, given the three vanishing points, the position
of the point S is absolutely fixed; there is only one
point at which the eye can be placed and find each pair
Fig. 28.   of vanishing points 90° apart. Fig. 28 illus-
trates this, the semicircles that contain the three right
angles being foreshortened into ellipses.

161. Another way of regarding the problem is to con-
sider that, since the plane in which each semicircle lies
is not perpendicular to the picture, but is inclined to it
at an unknown angle, the position of the station point
is really limited only by the condition that it must lie
somewhere in the surface of a hemisphere of which the
given trace is a diameter. As this is true for each of the
three traces, the station point must be a point common
to the three hemispheres. Now three hemispheres, whose
diameters form a triangle, can have but a single point in
common. Two of them will intersect each other in a
semicircle perpendicular to the plane of the triangle,
and the point where this semicircle is cut by the third
Fig. 29.   hemisphere will be the point in question. Fig.
29 illustrates this view of the subject.

162. It is plain from an inspection of the figure, and
of the little figure alongside, that the small semicircles in

which these hemispheres intersect will be projected as
straight lines at right angles to the lines connecting
their centres. But as the lines connecting   To find the
the three centres are obviously parallel to the   Centre.
three diameters, it follows that the three straight lines
in which these three semicircles are projected, and
which meet in the point C, the projection of the apex of
the pyramid, are drawn from the corners of the base per-
pendicular to the opposite sides. This affords an easy
method of determining the point C.

163. This proposition, that the projection of each
edge of the pyramid is perpendicular to the opposite
side of the base, is, in fact, merely an illustration of the
familiar proposition that if a line is normal to a plane,
its projection upon a second plane intersecting the
first is perpendicular to the line of intersection. Each
edge of the pyramid is obviously normal to the oppo-
site face of the pyramid, and its projection upon the
base must accordingly be perpendicular to the oppo-
site side of the base, where the face of the pyramid
cuts it.

164. *Note.* This is not the place to demonstrate the proposition, of
which the demonstration is to be sought in the treatises on plane geo-
metry, that perpendiculars let fall from the vertices of a triangle upon
the opposite sides will meet at a point. But it is worth while, perhaps,
to observe that this point of symmetry within the triangle is only one
of four such points, the others being (*a*) the centre of the inscribed
circle, (*b*) the centre of the circumscribed circle, and (*c*) the centre of
gravity. Fig. 30 *a*, *b*, *c*, *d*, exhibits a comparative view of   Fig. 30.
these four points.

165. The point C being thus ascertained, it only re-

mains to determine the height of the pyramid, that is, the distance of the station point, S, in front of the pic-
To find the ture, and the length of the three edges of the
Station Point. pyramid, that is to say, the distance of the three vanishing points from the station point.

Fig. 31 shows how these distances may be determined. A plane perpendicular to the picture is passed through either edge of the pyramid. Its intersection with the opposite face and with the plane of the picture, or base of the pyramid, will form a right-angled triangle. This triangle, when revolved about its hypotenuse into the plane of the picture, will give V S, the length of the edge in question, and the height of the pyramid, or dis-
Fig. 27. tance of the eye from the picture, C S. This operation is repeated on a larger scale in Fig. 27, giving S_1.

166. Fig. 28 shows how the distance of the eye from
Fig. 28. two vanishing points, that is to say, the length of the two edges of the pyramid, can be found at once by revolving one of its triangular faces into the plane of the picture. Each semi-ellipse becomes a semicircle, on the circumference of which is found the station point in its revolved position at D, and D A and D B are the length of two of the edges.

167. Fig. 32 exhibits the curious geometrical rela-
Fig. 32. tions that result from the application of this
Geometrical process to all three faces at once. It will be
relations. noticed that the two semicircles that start from each vanishing point meet and intersect on the opposite trace, just at the point where the perpendicular

drawn from the vanishing point in question through the centre, C, strikes it. If now, from each vanishing point as a centre an arc be drawn with a radius equal to the distance of that vanishing point from the station point, each arc will be the *locus* of its point of distance, and the intersection of these arcs with the traces will give the six points of distance sought. Moreover, not only will each of these arcs pass through two out of the three revolved positions of S, but its points of intersection with the other two arcs will lie in the perpendiculars let fall from the other two vanishing points upon the opposite traces.

168. This last observation gives the means of determining all six points of distance by revolving into the plane of the picture only a single one of the faces of the pyramid, as is illustrated in Fig. 33. If the triangle V^M S V^O, right-angled at S, is revolved around $V^M V^O$, S will fall at D, and the points of distance D^M and D^O, two of each, are easily determined, $V^M D^M$ being equal to V^M D, and $V^O D^O$ to V^O D. But the *locus* of D^L passes through the point where the arc D^M D D^M cuts V^O C, and also through the point where D^O D D^O cuts V^M C. $V^L D^L$, then, is easily determined, and the two points D^L ascertained without further labor.

To find the six Points of Distance.

The several points of distance in Fig. 27, to which Fig. 33 is similar, are obtained in this way. Fig. 33.

D, which is D^R, since D V^R is obviously equal to S V^R, enables us to determine another D^R just below V^O.

169. The phenomena of intersecting planes, with the vanishing points of their lines of intersection at the intersection of their traces, are the same in Three-Point as in Two-Point or in One-Point Perspective, and are again and again illustrated in the plate.

CHAPTER IX.

THE PERSPECTIVE OF SHADOWS.

170. THE rays of the sun, being practically parallel, constitute a single system of parallel lines, The pheno-mena of shadows. with the same two vanishing points, 180° apart (5). Both these vanishing points may be found by looking in the direction followed by the rays. If one looks up in the direction of the rays, he of course sees the sun itself. If he looks down, away from the sun, he sees the shadow of his own head. Of the two vanishing points of the system, then, one is in the sun itself, and the other, just opposite, is in the shadow of the spectator's head, and is, of course, as far below the horizon as the sun is above it.

It sometimes happens in photographs that the shadow of the camera is seen in the foreground, at the vanishing point of shadows.

171. If the sun is in front of the spectator, it is the first of these vanishing points, that in the sun The position of the sun. itself, which is behind the plane of the picture, as in Fig. 34, and the vanishing point of the Fig. 34. sun's rays, V^s, is above the horizon. If the sun is behind the spectator, as in Fig. 35, the other Fig. 35. vanishing point is in the plane of the picture,

and V^s, which we now call the vanishing point of shadows, since every point throws its shadow towards it, is below the horizon. It appears in the figure in the extreme right-hand lower corner of the plate, beyond

Fig. 36. If the sun is just in the plane of the picture, neither in front of the spectator nor behind him, but on one side and above, the light falls parallel with the plane of the picture, and both vanishing points are at an infinite distance upon that plane. This is illustrated in Fig. 36.

Fig. 36.

172. The shadow of every point is accordingly a line proceeding from that point, through the air, towards the vanishing point of shadows. This line is generally invisible, the air being generally transparent; but when the air is loaded with dust or moisture this line of shadow becomes visible, as is often witnessed at sunset, when the shadows of clouds near the western horizon are thrown across the sky in parallel lines, — lines which seem to converge towards the sun in the west, and in the east to converge towards the vanishing point of shadows opposite the sun. If this invisible shadow of a point strikes any solid object, it becomes visible as a point of shadow on its surface. The *invisible* shadow of a point, then, is a line in space ; the *visible* shadow of a point is a point situated where the line of invisible shadow pierces any intercepting surface upon which the shadow may fall.

The shadow of a point.

173. In like manner, the *invisible* shadow of a line is a surface in space, and the *visible* shadow of the line is a line, being the line in which this

The shadow of a line.

surface intersects the surface upon which the shadow
falls. If the line that casts the shadow is curved, the
invisible shadow is cylindrical; if it is a straight line
that casts the shadow, the shadow in space is a plane;
and if the surface that receives it is also a plane, the line
of visible shadow is a straight line, being the line of in-
tersection of two planes.

In Fig. 37 both the visible and the invisible shadows
of a line are represented. The sun is supposed
to be behind the spectator, in such a position
Fig 37.
that the shadow of the spectator's head is thrown upon
the ground within the limits of the picture at V^s. The
rays of light and the shadow of every point in the line
are directed towards this point. The shadow in space is
seen to be a plane, and the shadow on the ground and
steps is seen to be the intersection of this plane with
the several planes which it encounters.

174. If a solid body casts a shadow, the *invisible*
shadow, passing downward through the air
away from the sun, is a solid cylinder, or solid
The shadow
of a solid
body.
prism, according as the line upon the body that casts the
shadow is a curved line or rectilinear. This line is ob-
viously the line upon the surface of the body which
separates the side towards the sun, which is in light,
from the shady side. This line is called the *dividing
line of light and shade.* The *visible* shadow of the solid
object, seen upon any other object, is a surface, the shape
of which is determined by the shadow cast by the line
of light and shade.

To find the shadow of a solid body, then, is the same

thing as to find the shadow of a line, namely, the shadow of its dividing line of light and shade.

175. In finding the shadow of a point, also, the only practicable way is first to find the shadow of some line that passes through the point, and then to find in this line of shadow the point of shadow that corresponds to it. This point is easily found by drawing a line, representing the invisible shadow of the point, through the air, from the perspective of the point to the vanishing point of shadows, V^s. Its point of intersection with the visible shadow of the auxiliary line is the shadow of the point in question.

Thus in Fig. 37 the point A has its shadow at a; and, conversely, the shadow at b, at the bottom of

Fig. 37.

the steps, is cast by the point B. This shows just how much of the stick throws its shadow on the

ground. In Fig. 35 the shadows of all the

Fig. 35.

principal points, such as the top of the sign-post, or of the peak of the gable, are found by drawing lines to V^s, and marking their points of intersection with the shadows of the lines in which these principal points lie.

176. The whole problem of shadows thus resolves itself into the problem of finding the shadow

The visible shadow of a line: the intersection of the plane of its invisible shadow with the plane on which it falls.

of a line; and as in this paper we shall consider only the case of straight lines throwing their shadows upon plane surfaces, we have to do only with rectilinear shadows, lying where the plane of the invisible shadow of the line

cuts the plane of the surface that receives it. The whole question becomes, then, simply a question of the intersection of planes.

177. Now, the line of intersection of two planes, as we have seen in the case of two intersecting roof planes, has its vanishing point at the intersection of the traces or horizons of those planes (34). Hence the (visible) shadow of a line upon a given plane will have its vanishing point at the intersection of the trace of that plane with the trace of the plane of the (invisible) shadow of the line. And since, if any plane is given in perspective, its trace is already known, the only thing that remains to be done is to find the trace of the plane of the shadow. The direction of the line and the direction of the light are, of course, also given; that is to say, their vanishing points also are known.

178. But these two vanishing points being known, the trace of the invisible plane of the shadow is easy to ascertain. For the trace of a plane passes through the vanishing points of any two lines that lie in it (13, II.). Now, as may be seen in Fig. 37, the line that casts it lies in the plane of the shadow, and so does the invisible shadow of any point in that line. The trace of the plane of the shadow accordingly passes through the vanishing point of the line that casts it, and through the vanishing point of shadows V^s, and may be found at once by drawing a line through them.

The horizon of the plane of the invisible shadow.

Fig. 37.

179. Thus in Fig. 34 the trace of the shadows of the right-hand horizontal lines R, whose vanishing point is

V^R, is the line T S R, the trace of the shadow of R,
joining V^R and V^S. In the same way, if we
call the plane of the shadow of L, S L; that
of M, S M; that of Z, S Z, etc., we shall have
of M, S M; that of Z, S Z, etc., we shall have
the traces of these planes, T S L, T S M, T S Z,
etc., connecting V^S with V^L, V^M, V^Z, etc., respectively.
As V^Z, the vanishing point of vertical lines, is at an in-
finite distance in the zenith, T S Z, like T R Z and T L Z,
is a vertical line.

Fig. 34. The
sun in front
of the spec-
tator, and
behind the
picture.

It is not in general very easy to follow these invisible
planes of shadow in imagination, and to understand just
how they go, by a mere inspection of the figure. But in
the case of a vertical line, such as that of the nearest corner
of the building, one can see that the plane of the shadow
must be a vertical plane nearly parallel with the right-
hand side of the house, but not quite so, being at a less
angle with the plane of the picture. It seems reason-
able, then, to find its trace T S Z parallel with T R Z and
a little further to the right.

180. Since all these planes of shadow have one ele-
ment parallel to the light, all their traces, as
is seen both in Fig. 34 and in Fig. 35, pass
through V^S. This point thus furnishes an
illustration of the proposition (13, *b*) that the traces of
all the systems of planes that can be passed through
a line, or drawn parallel to it, in any direction, pass
through the vanishing point of the system to which
the line belongs, and intersect each other at that
point.

Fig. 35. The
sun behind
the specta-
tor.

V^S, accordingly, resembles the centre of a wheel, the

spokes of which are drawn through the vanishing points of all the lines in the picture.

181. The intersection of the traces of these planes of shadow with the traces of the different planes on which the shadows fall gives the vanishing points of the different lines of visible shadow. Thus in Fig. 35 the trace of the shadow of the sign-post is T S Z ; and the successive portions of its shadow which fall upon the ground, R L, upon the side of the house, L Z, and upon the roof, L M, are directed to the points of intersection of T S Z with T R L, or the Horizon, with T L Z and with T L M respectively. *The vanishing point of the line of the visible shadow.* *Fig. 34.*

Fig. 38 exhibits these relations in a diagram. The vanishing points of shadows are marked in this plate by four letters, thus, V^{SZ-LM}, which signifies the vanishing point of the shadow of vertical lines falling upon the plane L M. It hardly needs to be pointed out that all the different shadows cast by the lines of any system, on whatever plane they fall, have their vanishing points on the trace of the shadow of that system ; and that all the shadows that fall on a plane, whatever kind of line casts them, have their vanishing points in the trace of that plane. *Fig. 38.*

182. The direction of the lines of shadow being thus predetermined by the determination of their vanishing points, and their length being fixed either by the limits of the plane on which they fall or by the limits of the length of the lines that cast *The initial point of a shadow.*

them, everything is known about them except their
exact position.   To fix this it is necessary to know the
position of some one point in the line of shadow.   This
is generally given in the conditions of the problem.   In

Fig. 34.

Fig. 35, for example, so much of the shadow
of the sign-post as falls on the ground is deter-
mined in position by its initial point.   The shadow be-
gins where the pole touches the ground.   Thence it
goes off in the direction of its vanishing point, at the
intersection of T S Z with the horizon, as far as the
ground extends; that is, to the wall of the house.
The terminal point of the shadow on the ground is
the initial point of the portion that runs up the wall,
and so on.

All the shadows in Fig. 34 and Fig. 35 are drawn in

Figs. 34, 35.

this way, and illustrate these principles.   It
is not worth while to take space to describe in
detail what may now easily be understood from an in-
spection of the plate.

183.   If no convenient spot to begin at is furnished by
the conditions, it is necessary either to prolong the line
and extend the plane until they meet, in which case
the initial point of the shadow is the point at which
the line pierces the plane, or to pass an auxiliary line,
in any direction that is most convenient, through some
point in the given line, and to find the point where it
pierces the given plane.   This point will be the initial
point of the shadow of the auxiliary line; the shadow
of the point selected can then be determined upon it,
and the shadow of the given line drawn through that

point of shadow.   The auxiliary line must always be employed when the line that casts the shadow is parallel, or nearly parallel, to the plane on which it falls.

It is generally most convenient to take this auxiliary line in a vertical direction.   This is done in the case of the balloon, shown in Fig. 35.   Its position being known, a line can be dropped from it upon the plane beneath, and the shadow of the balloon drawn, at the end of the shadow of the line.

184. Fig. 36 illustrates the case in which the sun is neither behind the spectator nor behind the picture, but just in the plane of the picture, throwing his rays parallel to it and to the plane of measures.   This is by far the most convenient position for the sun when the objects represented are drawn in angular or two-point perspective, as they generally are.   It is almost sure to produce a picturesque disposition of light and shade.

Fig. 36. The sun in the plane of the picture.

It is also much simpler and easier to work than either of the other cases.   For since the vanishing point of shadows is at an infinite distance, V^s is entirely off the paper, and the rays of light cross the paper at their real inclination with the ground; and not only the lines of invisible shadow, but the traces of the planes of shadow, have the same inclination.   There is an apparent difficulty in the case of vertical lines, and of other lines parallel to the picture, since their vanishing points, as well as the vanishing point of shadows, are at an

infinite distance, and it is impracticable to find the
trace of their shadows by drawing a line from one in-
finitely distant point to another.   But it is obvious that
these lines must cast their shadows in planes parallel
to the picture.   The shadow of such a line on any
plane, then, will be parallel to the trace of that plane
(80).

Fig. 36 furnishes abundant illustration of this case.

185.   It is not always quite obvious, from mere in-
spection of a drawing, which of the edges of a
solid object really determine the form of its
shadow ; which of its lines go to make up the
dividing line of light and shade (174) ; which of its sur-
faces, that is, are turned towards the sun, and which
are turned away from it.   It is not easy to tell, for ex-
ample, whether the farther slope of a roof is in the light
or not ; whether the eaves or the ridge is casting a
shadow on the ground beyond.   Conversely, it is not
always easy to judge just where the sun must be put in
order to produce the distribution of light and shade upon
the different surfaces that is desired.

*The dividing
line of light
and shade.*

These difficulties disappear, however, if we consider
that what we want to know is whether or not
the sun has set, so to speak, to the plane in
question, and apply to that plane the same test that we
apply to the horizontal plane of the ground.   If the sun
is above the horizon, or the vanishing point of shadows,
opposite the sun, is below the horizon, we know that the
ground is in light, and *vice versa*.   So of every other plane :
if the sun is above its horizon, it is in light ; if the sun

Sunset.

has set to it, and the vanishing point of shadows is above its horizon or trace, the plane is in shade.

In Fig. 35, for example, V^s is beyond T R Z ; it is above the horizon of the plane R Z, the right-

Fig 35.

hand side of the house. This side of the house is accordingly in the shade ; the sun has set to it. If V^s were moved to the other side of T R Z, below this horizon, that side of the house would obviously be in the light. So of L M′, the plane of the back of the roof. V^s is above T L M′; the sun has set to that plane also, and the dividing line of light and shade runs along the ridge ; it is the ridge, not the eaves beyond, that casts a shadow.

186. But it is to be noticed that when the sun set to the end of the house which is in sight it rose to the other end of the house which is parallel to it, and as both these planes have T R Z for their trace we must discriminate between them.

This we can do if we recall the distinction already pointed out between the surfaces that are in sight and those that are not : " A plane surface upon a solid object cannot be seen unless it is on the side of the object next the trace of the plane " (12) ; that is to say, unless it is below its horizon (38).

187. Bearing this in mind, we have the following rule for the illumination of surfaces by the sun : —

A plane surface that is in sight, being turned towards its trace, is in the light if the sun is on the farther side of its trace, or if the vanishing point of shadows is on the hither side.

A plane surface that is out of sight is in the light if
the sun is on the hither side of its trace, or the vanish-
ing point of shadows on the farther side.

188. It has not seemed worth while to encumber the
figures with constructive lines. It is for the
Notation.
most part left to the intelligence of the reader
to trace, point by point, the application of these prin-
ciples in the various cases they present. In Fig. 35,
however, a notation has been used for the outline of
some of the principal shadows which will serve
Fig. 35.
both to recall the principle of their construc-
tion and to indicate the point to which they are directed.
The expression "S Z on R L," for instance, indicates
that the outline to which it is attached is the shadow of
a vertical line, Z, on a horizontal plane, R L; "S N on
L M," in like manner, when applied to the shadow upon
the upper roof of the iron rod which supports the chim-
ney, signifies the shadow of a line N upon the plane L M.
In both cases the line of shadow is a line of intersection
of two planes, and has its vanishing point at the inter-
section of their traces; in the former case at $V^{SZ.\ RL}$,
where T S Z intersects with T R L, in the latter case
at $V^{SN.\ LM}$, where T S N meets T L M.

189. In a few cases the lines of invisible shadow have
been indicated, converging to V^S, to show their use in
determining the length of the visible shadow. In Fig.
34, where this is done, it will be noticed that
Fig 34.
the dotted lines drawn from the top of the
posts converge at the sun, while their shadows converge

to the point on the horizon below the sun. In Fig. 36 the visible shadows of vertical lines are parallel and horizontal, while the dotted lines Fig. 36. that indicate the invisible shadows follow the real direction of the light, falling parallel to the picture, and are parallel to each other, and also to all the traces of shadows drawn through the various vanishing points.

190. It will be observed that wherever a line is parallel to the plane on which it casts its shadow it is an element of both systems of planes ; the traces of both planes accordingly pass through its vanishing Shadows on planes parallel to the lines that cast them. point (13, b), which is their point of intersection, and the shadow is parallel to the line that casts it, as it should be, having the same vanishing point. This is illustrated in Fig. 35 by the shadows cast by vertical lines upon the vertical planes, by horizontal lines upon the ground, and by the inclined lines M upon the inclined planes L M.

The shadows cast by artificial light are discussed in Chapter XIV., where will also be found, in section 313, an alternative method of finding certain shadows cast by sunlight.

CHAPTER X.

THE PERSPECTIVE OF REFLECTIONS.

191. Let us now consider how things look in a mirror, — whether in an artificial mirror, or looking-glass, or in the natural mirror formed by the surface of still water.   The question is obviously a little more complicated than those we have been discussing, inasmuch as a new element is introduced. We have now to consider not only the position of the spectator and the position of the picture, and their relation to the position of the object reflected and to the mirror that reflects it, but also the relation of the object reflected and of the reflecting surface to each other.   Either of these may be parallel, perpendicular, or inclined to the others.

Given, the position of the spectator, that of the picture, that of the object, and that of the mirror, it is required to depict not only the lines and surfaces of the object itself, with all their vanishing points and traces, but the reflection of the object in the mirror, and the vanishing points and traces of the reflection.

192. In reflections, however, as in shadows, and as everywhere in perspective, the various problems of the point, the line, and the surface are all comprised in the problem of the line (205, 206).   How to draw a given

line through a given point is the only question.   For the perspective of a point can be got only by finding the perspective of a line, or of two lines, passing through it, and a surface is drawn in perspective by drawing the perspective of the lines that enclose it.

The problem of reflections, then, is this : to draw the reflection of a given line in a given mirror.   The position of the centre of the picture, C, is, of course, known, and the distance of the eye at the station point, S, in front of it ; and the vanishing point of the line and the trace or horizon of the plane of the mirror ; with the position of some point in the line and of some point in the mirror.

193.   Let us first take the most general case, — that of a line inclined to the picture, at any angle, taken at random, reflected in a mirror which is inclined to the plane of the picture and to the horizontal plane, in any position, taken at random.

The general case : the reflection of an obliquely inclined line in an obliquely inclined mirror.

This disposition is shown in Figs. 39, 40, and 41, which illustrate three successive steps in the solution of the problem.

Plate IX., Figs. 39, 40, 41.

In each of these is shown two sides of a room, making an angle with the plane of the picture, on one of which a mirror hangs at an angle with the wall.   The plane of this mirror we will call, in pursuance of the system of notation adopted in these papers, the plane L M′, since its horizontal element is obviously parallel to the edge of the floor, whose vanishing point is V^L, and its line of steep-

est slope descends to the right, in the general direction which we have called M'. $V^{M'}$ will be below V^R, and T L M', the trace of the plane of the mirror, will pass through $V^{M'}$ and V^L, as shown. The position of the mirror is fixed by that of its lower edge where it intersects the plane of the floor.

The line whose reflection in this mirror we are to find slants upwards to the left nearly in the direction we have called N; but as it is not parallel to the plane L Z, we will call it O, its direction being given by its vanishing point V^O. Its position is fixed by that of its nearest point, whose distance above a point on the floor is shown.

Let us also call the direction of lines at right angles to the mirror — that is to say, perpendicular or *normal* to it — by the letter T, *norma* being Latin for T-square. They will be parallel to the *axis* of the mirror, and their vanishing point will be V^T.

194. It is obvious from the inspection of either figure that O', the reflection or *image* of the line, will look like just such another line as far behind the surface of the mirror as the line itself, or object, is in front of it; and that a line drawn from any point in the object to the corresponding point in the image — that is to say, from any point to the reflection of that point — will be normal, or perpendicular, to the mirror, and will have its vanishing point at V^T.

All these normal lines together, moreover, make up a normal plane, also at right angles to the mirror, which may be called the plane O T. And as the trace of any

plane passes through the vanishing points of all the
elements of the plane, T O T, the trace of the normal
plane O T, passes through V^o, the vanishing point of the
given line, and through V^T, the vanishing point of nor-
mals.

And as, conversely, every line that lies in a plane has
its vanishing point in the trace of the plane, O', the
image of O, must have its vanishing point $V^{o\prime}$ also in
the trace T O T.

Finally, it is plain that the line I, in which this nor-
mal plane intersects the plane of the mirror, seems to
lie equidistant between the given line O and its image
O', and to bisect the angle they make with each other ;
and that since it lies at the intersection of these two
planes, its vanishing point must be at V^I, the intersec-
tion of their traces.

195. The problem of reflections is solved when the
line I is fixed and the point V^T ; for the image or reflec-
tion of every point in the given line lies on a normal
passed through the point, at a distance beyond the line I
equal to that of the point itself on the hither side of it.
This equal distance can be obtained, as shown, by means
of the method of triangles, using a line of measures, $o\,t$,
drawn parallel to T O T, and an auxiliary vanishing
point, at V, as a point of proportional measures. By
taking $o\,t$ of any convenient length, and then taking $t\,o'$
equal to it, the initial point of O' upon the line T is
easily ascertained. But for a complete solution of the
problem, it is necessary to determine also $V^{o\prime}$, the vanish-
ing point of the image.

Fig. 39 shows how V^T, and, consequently, V^I, are de-
termined; while Fig. 40 and Fig. 41 show
how I itself and $V^{O\prime}$ are obtained.

196. The first problem is how to find V^T, T L M
being given; that is to say, given a plane or system of
planes by its trace, to find the vanishing point of the
axis of the system, *i. e.*, its vanishing point of normals.

The solution of this problem depends upon the prin-
ciple already illustrated in Fig. 29, Plate VII., that "if
a line is normal to a plane its projection upon a second
plane intersecting the first is perpendicular to the line of
intersection." For if (Fig. 39) we pass through
the station point, S, in the air, at a distance in
front of C equal to the line $C S_1$, a plane parallel to the
mirror, it will intersect the plane of the picture in the
The vanish- line $T L M'$, and if we pass through the same
ing point of
the lines nor- point a line normal to that plane and parallel
mal to the
mirror. to the lines normal to the mirror, it will
pierce the plane of the picture at V^T; for if from the
station point one looks in a direction parallel to the
lines of any system he will see the vanishing point of
that system. Now, since the projection of the station
point on the plane of the picture is at C, the projection
of this line will be $C V^T$, and this line will be perpen-
dicular to $T L M'$ at the point *a*. *A line, then, drawn
through C perpendicular to the trace of any given plane,
will pass through the vanishing point of lines normal to
that plane.*

197. If now a line be drawn from the station point

S, to the point a, it will be at right angles to the line drawn from the station point to V^T. $a\,S\,V^T$ is accordingly a right-angled triangle of which $a\,V^T$ is the hypothenuse, and if this triangle be revolved into the plane of the picture about $a\,V^T$, S will fall at S_2, ($C\,S_2$ being taken equal to $C\,S_1$), and a line drawn at right angles with $a\,S_2$ will give V^T at its intersection with $a\,C$ prolonged.

198. V^T being thus ascertained, T O T, the trace of the normal plane, is drawn through V^T and V^O, and V^I is obtained at its intersection with T LM' (194).

The trace of the normal plane.

199. Fig. 40 shows how the line I, at the intersection of the normal plane with the mirror, is determined in position; its vanishing point V^I being already found, it is necessary only to determine one point in the line I, upon the surface of the mirror. The line can then be drawn through this point and the point V^I.

Fig. 40.

The point where the given line pierces the mirror.

But it is plain, from an inspection of Fig. 39, that if the line O were prolonged until it touched the mirror its reflection O' and the line I lying between them would be prolonged also, and that all three lines would meet at the point where O pierced the surface. This point, then, would be a point of the line I such as we are seeking.

The problem resolves itself, then, into that of finding where a line pierces a plane, both being given in perspective.

200. To find where in the figure (Fig. 40) the line O pierces the plane of the mirror L M′ we pass a vertical plane, O Z, through the line O; its trace is T O Z. The dotted line in which this plane intersects the floor has its vanishing point at the intersection of the Horizon T R L, the trace of the plane of the floor, with T O Z, the trace of this vertical plane (13, III.) The point e is a point common to the plane L M′ and to the plane O Z; that is to say, it is one point of the line in which the vertical plane through O intersects the surface of the mirror, the point f, where the traces of these planes intersect, being the vanishing point of this line of intersection. The line fe prolonged is then this line of intersection itself, and the point g, where the line O prolonged meets the line fe, is the point where it pierces the mirror.

201. But the point g, being also a point of the line I prolonged, a line drawn through g and the vanishing point V^I gives the indefinite line I which we are seeking. Normals drawn through the extremities of the line O to V^T cut off from the indefinite line I the finite portion required. This is shown in Fig. 41.

202. Fig. 41 also shows how $V^{O′}$, the vanishing point of
 the reflection of a given line, may be obtained,
Fig. 41. The Vanishing Point of the reflection. V^T, V^I, and I having already been determined, and O and V^O being given.

It is plain that O, I, O′, and T, in Fig. 39, are all in the same normal plane, and that if from the position of the eye at the station point, S, in the air, in front of C,

lines are drawn to the vanishing points V^O, V^I, $V^{O\prime}$, and V^T, these lines will lie in the plane which, passing through the eye, intersects the plane of the picture in the trace T O T. The lines S V^T, S V^O, S V^I, and S $V^{O\prime}$ will all lie in the same plane, and will be parallel respectively to T, O, I, and O′, and will make the same angles one with another. But since I bisects the angle made by O and O′ (194) so must S V^I bisect the angle at S, made by S V^O and S $V^{O\prime}$. If, then, in Fig. 41 we revolve the plane triangle V^T, S V^I, right-angled at S, around its hypothenuse, into the plane of the picture, the trace T O T and all the vanishing points upon it will remain where they are, and S will fall at S_4. For the real distance from the station point to the trace T O T at b is b S_3, the hypothenuse of the right-angled triangle of which C b is the base and C $S_3 = $ C S $= $ C S_1 the altitude. This distance b S^3 laid off upon a perpendicular drawn through C to the trace T O T at b, gives the point S_4.

203. From this point lines drawn to the several vanishing points on T O T make the same angles one with another as do the lines O, I, O′, and T, and since I bisects the angle between O and O′, $V^{O\prime}$ is easily ascertained by drawing a line on one side of S_4 V^I at the same angle that S_4 V^O already makes upon the other side of it.

204. In these figures the mirror stands at an angle with the plane of the picture, and also at an angle with the ground. The case is somewhat simpler when the latter angle is 90°, the

The mirror vertical, but at an angle with the picture.

mirror being upright. The trace of the mirror becomes
vertical, and the normal lines are horizontal. If in Figs.
39, 40, and 41, we imagine the mirror set back into a
vertical position, it is clear that V^T will move so as to
coincide with V^R, S_2 with S_1, and T L M′ with T L Z;
f will be at an infinite distance, and the line $e\,g$ will be
vertical. But the essential conditions of the problem
will remain unchanged, and the trace of the normal
plane, the point where the given line pierces the mirror,
and the vanishing point of the image will be obtained
as above.

205. If the object reflected is a plane surface, its re-
The reflec-   flection must be found by finding the images
tion of a
plane figure.   of the lines that bound it, and their vanishing
points. These vanishing points will of course lie in a
straight line, the trace of the reflection of the plane;
this trace can be drawn as soon as the vanishing points
of any two of the elements of the plane are ascertained.
It is convenient to take one of these elements parallel to
the mirror (207).

206. To obtain the reflection of a point, a line must
The reflec-   be passed through it, and its image obtained as
tion of a
point.   above. The reflection of every point in this
auxiliary line, including the point in question, is then
easily found. But it simplifies the problem, as we shall
presently see, to take the auxiliary line either parallel
to the mirror or perpendicular to it (209).

207. For the reflections of lines which are parallel to

the mirror, being in every part as far behind the mirror as the lines themselves are in front of it, are Lines parallel to the mirror. also parallel to the mirror and to the line I, in which the normal plane in which they lie intersects its surface. They are accordingly parallel to their originals, and have the same vanishing points.

It follows that if a plane is parallel to a mirror, all the lines in it retain their vanishing points, so to speak, in the reflection, and the plane retains its trace, $i.\ e.$, the trace of the reflection is the same as the trace of the plane itself.

208. The images of lines perpendicular to the mirror also seem to retain their vanishing point, Lines perpendicular to the mirror. which is the vanishing point of normals, V^T. But it is more exact to say that the *other* vanishing point, 180° away, is reflected so that its image coincides with V^T. It follows that the reflections of planes perpendicular to a mirror have the same traces as the planes themselves; for the vanishing points of the parallel elements and of the normal elements are alike unchanged.

209. It appears, then, that *the reflections of lines and planes parallel or perpendicular to the reflecting surface have the same vanishing points and traces as their originals*, whatever the position of the reflecting surface.

210. The general problem of the reflections of lines and planes, whether parallel, perpendicular, or inclined to the mirror, having thus been discussed, it only re-

mains to consider the special cases in which the mirror
itself is parallel or perpendicular to the picture. In
both these cases the problem is a very simple one.

211. *When the mirror is perpendicular to the plane of*
the picture, as in Figs. 42 and 43, the normals
are parallel to the picture, their vanishing
point is at an infinite distance, and the trace

The mirror
perpendicu-
lar to the
plane of the
picture.

of the normal plane is at right angles to that of the
mirror. The trace of the plane of the mirror passes
through the centre, C ; the plane drawn through the
station point, S, intersecting the picture in this trace, is
perpendicular to the picture, and lines drawn from the
station point to the vanishing point of a line and to that
of its reflection strike the picture at equal distances
from this trace.

The vanishing point of the image of a line, then, in-
clined to the face of a mirror which is perpendicular to
the picture, *is as far on one side of the trace of the
mirror as the original vanishing point is on the other*,
on a line at right angles to the trace.

If a line is parallel to the picture, so that its vanish-
ing point is at an infinite distance, the image is inclined
to the trace of the plane of the mirror at an equal angle
on the other side.

212. If a plane is inclined to a mirror that is per-
pendicular to the picture, one element of the plane will
nevertheless lie parallel to it, and that element will re-
tain its vanishing point, which will lie in the trace of
the plane of the mirror (209) ; every other element will,
so to speak, shift its vanishing point to the other side

of the trace of the reflecting plane (211). *The trace of the image will then cross the trace of the mirror at the same point with that of its original,* making equal angles on the other side.

213. *When the mirror is parallel to the plane of the picture,* having the vanishing point of its axis at the centre, C, a line, or system of lines, in-clined to the mirror has the vanishing point of its image *as far from the centre,* C, *in one direction as that of the line or system is in the other,* on a line drawn through the centre. If a line goes up and to the right, its reflection will of course seem to go down and to the left at equal angles. The mirror parallel to the plane of the picture.

214. It follows that if a plane is inclined to a mirror set parallel to the picture the trace of its image *is parallel to that of the plane itself,* on the opposite side of the centre, C, and equidistant from the centre.

215. These points are illustrated in Fig. 42. The spokes of the spinning wheel are parallel to the right-hand mirror, the axle is perpendicular to it. All retain their vanishing points. Fig. 42.

The box on the left, with its cover, presents four systems of lines, two horizontal, L and R; two inclined, N and O. The reflection in the floor retains V^L and V^R, but exchanges V^N and V^O for V^{N1} and V^{O1}, on the opposite side of the Horizon (209, 211).

The reflection of the box in the second mirror, on the left, has for vanishing points V^{L2}, V^{R2}, V^{O2}, V^{N2}, across

the trace of the mirror T X A, the plane L Z_2 having T L Z_2 for its trace, inclined to T X A equally with T L Z (212).

The third mirror, parallel with the picture, in like manner gives V^{L3}, V^{R3}, V^{N3}, and V^{O3}, for the vanishing points of the main lines of the reflection of box (213).

216. The reflections are themselves reflected just like their originals, as in mirror No. 2. $V^{N1.2}$ is the vanishing point of the lid of the box, reflected first in the polished floor and then in the inclined mirror.

217. Fig. 43 illustrates these principles by the phe-

Fig. 43. Reflections in water. nomena of reflections in water, a mirror perpendicular to the plane of the picture.

The steps have their vanishing point at V^M and $V^{M'}$, their reflections at $V^{M'}$ and V^M, respectively.

218. It is to be observed that the phenomena of reflection enable us to determine the real distance of isolated objects, such as birds or distant mountains, the point on the plane of the water directly below them being midway between the object and the image. The distance of this point is the horizontal distance of the object.

It is also to be noticed that the different size and inclination of the sticks in the foreground, which are not very obvious in themselves, are made conspicuous by the difference of their reflections, as it would be indeed on solid ground by the difference of their shadows. These objects exhibit very clearly the relation of a line

and its reflection to the normal plane in which they lie, and to the line in which this plane intersects the mirror.

That the horizontal line of birds should be reflected in an inclined line, and the inclined line of birds in a horizontal line, is easily understood by observing the line of dots on the surface of the water, midway between the birds and their reflections.

Each bird, as well as each mountain top, and indeed every other point, has its reflection as far below the surface of the water directly beneath it as the point itself is above.   The plane of the surface of the water is supposed to extend, of course, beneath the land.

CHAPTER XI.

IN the ten preceding chapters we have considered all
the principal problems of plane perspective em-
braced in our scheme. That is to say, we have shown
how to obtain, upon a plane surface, the perspective rep-
resentation of a straight line ; whatever the position of
the surface, and whatever the position of the spectator,
we have shown how to obtain the position, magnitude,
and direction of the representation of a line, when the
position, magnitude, and direction of the line itself are
known. The problems of shadows and of reflections
have also been fully discussed, so far as concerns plane
surfaces. Throughout the whole investigation it has
been shown that the problem of the line includes the
problems of plane and solid figures and of the point.
In every case the vanishing point of every line and the
vanishing trace, or horizon, of every plane has been
ascertained, the solution being considered incomplete
until this was accomplished.

219. The only lines included in this survey have ac-
cordingly been right lines, and the only plane or solid
figures have been such as are bounded by right lines.
Any other line or outline in perspective, as elsewhere in
geometry, must in general be treated as a series of points,

the perspective representation of each point being ob-
tained separately.   But to this rule the circle, here
as elsewhere, constitutes an exception, its exceptional
importance making it worth while to give it special
consideration, while its peculiar geometrical properties
render the investigation exceptionally simple and easy.

We shall find also that the study of the circle in
perspective, and of its derivatives, the cylinder and the
sphere, introduces a new set of most interesting phe-
nomena, the investigation of which will, in the two
subsequent chapters, lead to theoretical and practical
conclusions of the first importance.

220. The perspective representation of a circle will ob-
viously be the line in which the plane of the
picture intersects a cone of rays of which *The perspec-
tive of a
circle, a conic
section.*
the vertex is in the eye of the spectator, at
the station point, and the base is the circle itself.   The
theory of conic sections establishes the fact that this
line of intersection will be a circle, ellipse, parabola, or
hyperbola, according to the angle at which the plane of
the picture cuts the cone of the rays, and this whether
the axis of the cone be at right angles to the circle or
inclined to it.   In other words, the cross-section of the
cone, perpendicular to its axis, may be either a circle or
an ellipse.   If the secant plane is parallel to the base, or
equally inclined to the axis in a contrary direction,
making what is called a *sub-contrary* section, the per-
spective will be similar in shape, though not of course in
size, to its original.

221.　This is illustrated in Fig. 44, Plate X., and also
by the three figures 54, 55, and 56, in Plates
XI., XII., and XIII., all of which show how
horizontal circles, whether below or above the eye, will
appear when viewed from different positions.

In Fig. 44 the spectator is shown as regarding the cir-
cular room on the right from three different positions.　At
S^A he is outside the room; the plane of the picture, $p\,a$,
cuts completely across the cones of rays, and the sections
are ellipses, as shown at A, below.　At S^B he is just upon
the edge of the circle; the plane of the picture, $p\,b$, cuts
the cones in a direction parallel to one side, and the sec-
tions are parabolas, as seen below at B.　At S^C the spec-
tator is fairly within the circle, and the intersection of
the vertical plane, $p\,c$, with the cones of rays gives
hyperbolas, as at C.　Figs. 54, 55, and 56,
which are reproductions of engravings of cir-
cular halls in the Vatican Palace, excellently illus-
trate these elliptical, parabolic, and hyperbolic lines:
the first being drawn, presumably with a *camera lucida*,
from a point outside the circle; the second, from a point
just on the edge of the room; the third, from a point
within it.

222.　Fig. 44 also illustrates the case in which the
cone of rays is intersected by the plane of the picture in
such a way as to give a sub-contrary section.　The small
horizontal circle forming the *eye* of the dome is the base
of a cone of rays which is cut by the plane $a\,p$, at an
angle with the axis of the cone equal to that made by
the circle itself, but taken in a contrary direction.　Both

are obviously angles of 45° with the axis, which is accordingly at 45° with the horizon. The perspective of the circle is accordingly a circle (220), as is shown below. In Fig. 54, also, Plate XI., the perspective of the circle at the top of the dome is almost exactly circular.

223. Fig. 45 illustrates more fully the sub-contrary section spoken of in the previous paragraph, and shown in Fig. 44. $A_1 A_1$ shows the circle

in its own plane, with its centre beyond the axis of the cone; $B_1 B_1$ shows its perspective in the plane of the picture, $p\,p$, with the centre of the cone of rays below its centre, and the point a, representing the centre of the original circle, lower still; $E_1 E_1$ shows the real shape of the cross-section E E, taken at right angles with the axis of the cone. This is an ellipse, whose centre coincides with the axis of the cone of visual rays, the centres of both circles being also given, one on one side and one on the other. The projections of the respective centres are shown in each case, at a, b, and e.

The line D D, parallel to A A, shows that a horizontal section of the cone taken at this place must be a circle like $A_1 A_1$; and since the sub-contrary section at B B is symmetrical with it, about the axis of the cone, it follows that $B_1 B_1$ also must be a circle.

224. It is to be noticed that the ellipse $E_1 E_1$ is the appearance that the circle would present from the station point, S. It would not appear as a circle, though its perspective is a circle. But neither does this perspective circle appear as a circle. It, too, is fore-

shortened into an ellipse in the sight of a spectator at S (230).

225. In fact, unless a circle is situated just at the centre of the picture, the ellipse which represents it in perspective is of a different shape from the ellipse which it presents to the eye. Horizontal circles, for instance, always present to the eye horizontal ellipses ; ellipses, that is to say, whose major axes are horizontal. But in perspective such circles, unless just above or below the centre, C, have their axes inclined, as we may see in Fig. 47. Yet these oblique ellipses, when seen from the proper position, the station point in front of C, are themselves apparently changed in shape by the effect of perspective, and foreshortened into horizontal ellipses.

Fig 47.

This seeming distortion in the perspective, which makes the outline in the drawing of a different shape from the apparent outline of the thing drawn, will form the subject of the next chapter.

226. In general, of course, the station point is out-side the circle to be represented, so that practically the problem of putting a circle into perspective is this : to find the ellipse which represents it.

The simplest and generally the most efficient way to do this is to suppose a square or octagon to be circumscribed about the given circle, at any angle that may be most convenient. The centres of these sides give, of course, four or eight points of the required ellipse. As the direction of the sides gives the

To draw the perspective ellipse.

direction of the ellipse at these points, it can easily be drawn with sufficient accuracy for practical purposes. If greater accuracy is required the number of sides of the circumscribing polygon can be increased.

This is illustrated in Fig. 47.   ·Fig. 47.

227. But in order to draw an ellipse with absolute precision it is necessary to find its centre, and the direction and length of its axes or principal diameters.

It is obvious from Fig. 44 and from Fig. 54, Plate XI., that the perspective of the centre of a circle Its centre does not coincide with the middle point of and extreme points. the ellipse, as indeed it cannot, since the farther half of a circle must appear smaller than the nearer half, and its radii shorter. Neither do the extreme points of the ellipse represent extreme points of the circle; the highest point in the perspective of an arch, as may be seen in the arched windows and niches of Fig. 54, is the perspective, not of the highest point in the arch, but of a lower point nearer the spectator. In fact, although a circle put into perspective appears as an ellipse, the diameters of the circle do not become diameters of the ellipse, but chords, which intersect at a point situated beyond the centre. On the other hand, the diameters of the ellipse, meeting and intersecting at its centre, are the perspectives of chords of the circle, which meet and intersect at a point within the circle nearer to the spectator than its centre. The tangents at the extremities of each diameter of the circle are parallel; but their perspectives of course converge to a vanishing point; and since, for each circle, all these

tangents lie in the same plane, these vanishing points all

lie in the same straight line, which is the trace or horizon of that plane. See Fig. 46.

228. These phenomena afford a curious illustration of certain well known geometrical properties of

the ellipse. If a point be taken at random anywhere within a circle or an ellipse, and chords be drawn through that point, then the tangents drawn from the extremities of each chord will have their point of intersection upon a right line. This line is called a *polar line*, the point assumed being called a *pole*. As a mere geometrical proposition this seems to have no special significance ; but the phenomena of perspective give it meaning. For a circle seen in perspective becomes an ellipse, its centre a pole, its diameters chords, and the polar line is the horizon upon which meet the tangents drawn from the extremities of its diameters.

229. Fig. 46 exhibits these relations, and illustrates

also the further proposition, which, however, does not seem to admit of similar interpretation, that if lines are drawn from the points where the tangents meet through the middle of the chords they will pass through the centre of the ellipse. This property we shall find a use for presently (235, 238).

230. Although in Fig. 44, A, and in Fig. 45, the small

vertical circle is the perspective of the horizontal circle at the top of the dome, so that, when seen from the station point, the two circles seem to coincide, yet, as we have seen (223), their centres do not

coincide, and neither of them coincides with the axis of
the cone. The centre of each circle appears as a pole of
the other. If the cone be cut by a plane at right angles
with its axis, as at E E, Fig. 45, the section will be an
ellipse, $E_1 E_1$, of which the axis of the cone will give the
centre, and of which the centre of the upper circle will
be a pole situated just below the axis, and the centre of
the lower circle will be another pole just above it. In
the horizontal circle the centres of the ellipse and of the
vertical circle become poles, and in the vertical circle
the centres of the ellipse and of the horizontal circle
become poles.

An ellipse, of course, may by perspective be fore-
shortened into a circle, — the centre of the ellipse be-
coming a pole, just as a circle appears like an ellipse.

231. It will be noticed in figures 54, 55, and 56,
and also in Fig. 47, that the vertical circles whose
centres are on the Horizon, and the horizontal circles
whose centres are exactly above or below the centre of
the picture, C, lie symmetrically on the paper; that is
to say, that their principal diameters, their major and
minor axes, are vertical and horizontal, but that other
ellipses have their axes more or less inclined.

Another and more comprehensive statement of this
phenomenon is this: that if a line drawn through the
centre of a circle normal to its plane, like the axle of a
wheel, crosses the centre of the picture, C, one axis of
the ellipse that represents the circle will coincide with
this line, and will also be directed towards C, and the

other will be at right angles to it.　Other circles, which do not, as it were, thus *face* the centre, C, will be projected in ellipses the direction of whose axes it is more difficult to determine.

Fig. 47 shows a number of circles, three of which, A A, B B, and D D, are vertical, and accordingly appear in plan as right lines, and two, E E and F F, are horizontal.　These appear in perspective at $A_1 A_1$, $B_1 B_1$, $D_1 D_1$, $E_1 E_1$, and $F_1 F_1$, respectively.　In all of these except the last, one of the principal axes passes through the centre, C.　In $F_1 F_1$, and also in the circles $A_2 A_2$ and $B_2 B_2$, the position of the principal axes is, so to speak, accidental.

232.　Let us first take the case of the ellipses which

Ellipses that face the centre. represent circles that *do* face the centre, C, and which accordingly lie symmetrically about a

Fig. 47. normal line joining this point with the perspective of their centres.　This line will, of course, also pass through the centre of the ellipse, as in $A_1 A_1$, $B_1 B_1$, and $E_1 E_1$, Fig. 47.

233.　If, as in $A_1 A_1$ and $B_1 B_1$, two of the sides of the circumscribing square or octagon are parallel to the plane of the picture, and their perspectives consequently parallel to each other and perpendicular to the normal line, the line joining their middle points will be the minor axis of the ellipse ; the major axis will cross it at its middle point, and it will only remain to ascertain the length of this major axis.

234.　Fig. 47 shows how this is done.　Let A A, B B,

and D D, in the plan, be three parallel and similar circles,
all touching the plane of the picture, and the last, as
appears from the perspective below, standing edgewise
to the spectator at S.   If the space between them were
filled up with other such circles, they would all together
constitute an elliptical cylinder, the apparent vertical
dimension of which would be the apparent height of
each of the circles and of the major axis of the ellipses
that represent them.   Now let the plane containing the
circle D D and the station point, S, be revolved into the
plane of the picture around the vertical line T R Z, in
which the two planes intersect.   S will, of course, fall at
the point of distance D^R; the circle D D will appear of its
true shape and size ; lines drawn from D^R tangent to the
circle D D thus revolved will determine the highest and
lowest points visible from S, and the points where these
lines cut the line T R Z will show the perspective of these
points in the plane of the picture, and fix the apparent
height of D D.   The circles B B and A A will appear to be
of the same height as D D, and $D_1 D_1$ will be the length
of the major axes of the ellipses that represent them.

235.   When, as in the case of the circle E E, the tan-
gent lines are not parallel to the picture, the square or
octagon that encloses the circle being in angular per-
spective, instead of being in parallel perspective, as in
the previous instance, the centre of the ellipse must be
obtained as above explained (229) and shown in Fig. 46,
by bisecting two of the chords, and drawing lines from the
vanishing points of their tangents.   The principal axes of
the ellipse may then be drawn, one towards the centre, C,

and the other parallel to the picture, that is to say, parallel to the horizon of the plane in which the circle lies (   ). The length *e e* of the latter, or major axis, may then be found by direct projection, as in the figure. 47

236. To find the length of the minor axis, the major axis and one point of the ellipse being given, it is only necessary to employ the usual device, shown in Fig. 48, founded upon the proposition that if a semicircle be erected on the major axis of an ellipse the distance of the different points of the ellipse from this axis will be proportional to that of the corresponding points of the semicircle. Thus, in the figure, one point *x* being given on the ellipse, and *x'* and *b'* obtained on the circle, the point *b* at the extremity of the minor axis is easily found, since the chords *b x* and *b' x'* meet on the line of the major axis prolonged.

Fig. 48.

237. When the ellipse does not lie opposite the centre, C, as is the case with $F_1 F_1$, and with $A_2 A_2$ and $B_2 B_2$, in Fig. 47, the normals drawn through the centres of these circles not passing across the centre of the picture (231), the only way to obtain the principal diameters, or axes, is first to obtain a pair of *conjugate* diameters. Conjugate diameters are diameters each of which is parallel to the tangents drawn through the extremities of the other. The axes are that pair of conjugate diameters which are at right angles with each other; and one is always the longest diameter that can be drawn in a given ellipse, and the other the shortest.

Ellipses that do not face the centre.

Fig. 47.

238. The quickest way to obtain a pair of conjugate diameters in oblique ellipses such as $A_2 A_2$, $B_2 B_2$, and $F_1 F_1$ is to construct first such horizontal and vertical ellipses as $A_1 A_1$, $B_1 B_1$, and $E_1 E_1$, respectively, opposite the centre, and to obtain their principal axes as just described. If now the centre of each oblique ellipse is found, as above (229), and lines passed through it perspectively equal and parallel to these principal diameters, they will be conjugate diameters of the oblique ellipses. They will not be at right angles, but each will be parallel to the tangents drawn at the extremities of the other; one will be parallel to the plane of the picture and the other perpendicular to it, and directed to the centre, C.

239. Conjugate diameters of the ellipses $A_2 A_2$, $B_2 B_2$, and $F_1 F_1$ being thus determined, their principal diameters or axes can now easily be obtained by the well known method of shadows, as in the figure.

240. Fig. 49, a, shows this ingenious device more in detail. It is called the Method of Shadows, because the ellipse is regarded as the shadow or projection of a circle. The process is this : —

Fig. 49, a.

A tangent being drawn at the extremity of one diameter parallel to its conjugate, a circle is erected also tangent at the same point, of such size that the ellipse might be its shadow. The shadow of every diameter of the circle will be a diameter of the ellipse, and the shadows of any two diameters of the circle which are at right angles with each other will be conjugate diameters

To obtain the principal axes of an ellipse when two conjugate diameters are given by the method of shadows.

of the ellipse, since the tangents at the extremity of one will be parallel to the other. The given conjugates of the ellipse are shadows of those diameters of the circle which are perpendicular to and parallel with the tangent line common to both circle and ellipse. Since the shadow of the diameter parallel to the tangent is also parallel to the tangent, the diameter and its shadow are parallel to each other and must be of the same length. This fixes the size of the circle, the distance of whose centre from the end of one diameter is equal to half the length of its conjugate.

241. It now only remains to find in this circle a pair of diameters at right angles to each other whose shadows will also be at right angles. But since it is plain that if these diameters are prolonged till they reach the tangent line their shadows will also be prolonged, and will reach the tangent line at the same points, the problem becomes a very simple one. It is only necessary to find two points on the tangent line which make right-angled triangles both with the centre of the circle and with the centre of the ellipse; that is to say, two points such that the portion of the tangent lying between them shall be the common diameter of two semicircles passing respectively through these two centres. The common centre of these semicircles must be a point on the tangent line equidistant from the two centres; a point easily found by erecting a perpendicular upon the middle of the line connecting them, as is done in the figure. Semicircles struck from this point c as a centre, with a radius equal to its distance from the centre of the circle or of the ellipse, give

the points a and b, through which diameters can be drawn in the circle whose shadows, drawn through the same points to the centre of the ellipse, are axes or principal diameters of the ellipse, both sets of diameters making right angles with each other.

242. As the centre of the ellipse is the shadow of the centre of the circle, the line that joins these centres may be considered to give the direction of the light, and lines drawn parallel to it through the extremities of the diameters of the circle will give the extremities of the corresponding diameters of the ellipse, or the length of the axes.

243. This operation, though long in the description, is simple in practice, and requires very few constructive lines, as is seen in Fig. 49, b, in Fig. 49, b. which the operation just described is repeated with no more construction lines than are necessary.

This method is used, as has been said, in finding the axes of $A_2 A_2$, $B_2 B_2$, and $F_1 F_1$, and all the necessary construction lines are given in the figure.

244. When only a portion of a circle is to be put into perspective it is generally best to construct the ellipse which represents the whole circle, and then to use so much of it as may be required. The perspective of arcs of circles, and of arches. This is eminently the case in sketching from nature or from the imagination, where it is difficult to determine the character of the perspective curve without aid from geometrical considerations. In drawing pointed arches, for instance, the character of the intersecting arcs

is best ascertained by completing the circles of which
they form a part, as in Fig. 50.   An inspection of the

Fig. 50.

figure shows that when the arch is above the
eye the nearer half is represented by the part
of an ellipse at which the curvature is the most rapid,
near the extremity of the major axis ; and the further
half by the flattest portion, near the extremity of the
minor axis.   When the circle is below the eye the nearer
part is the flattest.

This figure also shows that when a row of circles is
put into perspective their major axes are not parallel,
their inclination to the trace of the plane in which the
circles lie diminishing as they approach it.   This is
illustrated also in Fig. 53.

245. Figs. 51, 52, and 53 show three different ways

of drawing concentric circles.   Since concen-
tric circles have the same centre, the ellipses
which constitute their perspectives have of course the
same pole and polar line ; but the ellipses have not the
same centre, nor are their axes parallel.

246. The first method is shown in Fig. 51 ; it is ap-

Fig. 51.

plicable to the case where the perspective of a
circle is obtained by means of a circumscribed
square or polygon.   A second circle, concentric with the
first, is easily obtained by means of a concentric polygon,
as shown.

247. Fig. 52 shows how the second ellipse can be

Fig 52.

found when the first has been already deter-
mined in any way.   Let a line of measures be

drawn through the pole which is the perspective of the
centre of the circle, parallel to the polar line, or trace of
the plane in which the circle lies. If now any chord $a\,a$,
representing a diameter of the circle, be drawn through
this pole, and lines be drawn from its extremities to any
point V upon this trace or horizon, it will cut the line of
measures at two points, a' and a', whose distance from
the pole is the same. If now two other points, b' and b',
be taken, also equidistant from the pole, and lines be
drawn through them, the points b and b, in which they
intercept the same chord, will be points of an ellipse
which represents a circle concentric with the given cir-
cle, and as much smaller as $b'\,b'$ is smaller than $a'\,a'$.
In the figure the radius of the smaller circle is one half
the radius of the larger one, $b'\,b'$ being one half of $a'\,a'$.
It is obvious that since the lines meeting at V are par-
allel in space, the lines $a\,a$ and $a'\,a'$ are divided propor-
tionally. Any number of points can be obtained in the
same way as $b\,b$.

248. A third method of putting concentric circles
into perspective is shown in Fig. 53, — a fig-
ure which, like Fig. 47 in plate X, shows three
equal circles, A A, B B, and D D, lying in parallel planes
and equally distant from the picture, the last of which
stands *edgewise* to the spectator, so that it coincides with
a portion of T R Z, the trace of the parallel planes. In
the figure it is supposed that A A is the given circle,
concentric with which it is required to draw another
circle E E.

249. To effect this the circle D D is first found by

Fig. 53.

cutting off from T R Z a portion equal in height to A A, this height being measured above and below a line passing through the centres of the three circles. Parallel to this line let a second line be drawn through any point, 1, of the circle A A, to the corresponding point, 3, of the circle D D; and upon this line let any convenient point, as 2, between 1 and 3, be taken as the corresponding point of a third circle, B B. If now a fourth point, 4, be taken upon the line through the centres, the line 4 2 5 will be an element of a cone whose vertex is at 4 and whose base in the plane of the circle A A is a circle concentric with that circle. The intersection of the line 4 2, prolonged, with a radius of A A drawn through the point 1, fixes the point 5, in the circumference of the circle E E. By drawing other lines, parallel to the line 1 2 3, through other points of A A, any number of other points in B B and E E may now easily be obtained.

250. The radii of the circles E E and A A (or B B) are obviously proportional to the distance of the point 4 from the centres of E E and B B, and also to the chords drawn through the common centre of A A and E E parallel to D D, that is, to the trace T R Z. By changing the position of 4, the size of E E may be made larger or smaller.

251. It will be noticed, 1st, that this method not only gives the means of finding a larger circle concentric with A A, and lying in the same plane, but also of finding an equal circle, B B, lying in a parallel plane; 2d, that it makes no difference in what direc-

tion the axis of the cone passing through the three
centres is originally drawn, provided it is parallel to
the plane of the picture; and 3d, that this method is
as serviceable in drawing a concentric circle smaller
than the given circle as in drawing a larger one; for
if E E were the given circle a reversed process would
give A A.

252. As to the figures 54, 55, and 56, in Plates XI.,
XII., and XIII., it is perhaps worth while to   Figs. 54,
say that the excessive distortion apparent in   55, 56.
them is due simply to the fact that the station point,
or proper position of the spectator, is in each of them
within three or four inches of the page. This is within
the limits of distinct vision. But by looking through a
pin-hole in a card the prints can be distinctly seen when
held even at the end of one's nose; and when so viewed
it will be seen that not only the ellipses in Fig. 54, but
the parabolas and hyperbolas in Figs. 55 and 56, look
like circles, as they should. The apparent distortion
entirely disappears.

CHAPTER XII.

DISTORTIONS AND CORRECTIONS. — THE HUMAN FIGURE.

253. It has been pointed out in the previous chapter that the perspectives of circles often look very queer; the ellipses by which they are represented seem unaccountably and even unnaturally inclined, their principal axes slanting in directions difficult to anticipate. The effect of this is particularly objectionable when the circle forms the base of a cylinder or when it is horizontal. The base of a cylinder always presents the appearance of an ellipse whose major axis is at right angles with the axis of the cylinder, and it is offensive to find it drawn otherwise, as in perspective often happens.

254. A horizontal circle always appears to the eye as a horizontal ellipse, as an ellipse, that is to say, whose major axis is parallel to the horizon and whose minor axis is perpendicular to it, and it is extremely unpleasant to see it drawn with the axes inclined. This is illustrated in Plate XIV., Fig. 58, by the perspective plan of the capital at the top of the figure and that of the base at the bottom. The effect of this would be so disagreeable if the curves of the capital and base were inclined in

Plate XIV.

Fig. 58.

Horizontal circles.

like manner, that it is customary to introduce a certain
correction, as it is called, as is done in the figure. These
lines are accordingly drawn as horizontal ellipses, just as
if these objects faced the centre of the picture (231).

255. Fig. 59, *c*, still further illustrates this point,
showing that in the column at the centre of
the picture the ellipses are horizontal, and
that the others are more and more inclined
as they are farther removed from it, which
looks like an unnatural distortion. In this figure, more-
over, the outer columns, which as seen from the station
point at S would look the smallest, since they are far-
thest from the eye, are on the contrary drawn larger in
diameter, an apparent distortion even more offensive
than the other. So with the spheres by which the
columns are surmounted. The outline of a sphere al-
ways looks like a circle; it is not agreeable to find it
drawn as an ellipse. But in perspective it must always
be an ellipse, unless its centre is just at the centre of
the picture; for the perspective representation of the
sphere is the section of a right cone with a circular
base, the base of the cone being that great circle of the
sphere which separates the side of the sphere one can
see from the further half of the sphere that he cannot
see; it must always be an ellipse unless the axis of the
cone is perpendicular to the plane of the picture.

256. Of course all these distortions disappear when
the eye is at the station point, at a proper distance in
front of the picture, opposite the centre C. From that
point of view the perspective lines exactly cover and

Fig. 59, c.

Distortions
of circles,
cylinders,
and spheres.

coincide with the oulines of the objects. But practically it is impossible for the spectator always to be exactly at the station point, and since from every other point, circles, cylinders, and spheres appear, in general, to be more or less distorted in the manner we have just seen, it is customary here also to apply corrections. These corrections are palpable violations of the rules of perspective made in order to avoid the disagreeable con-

Corrections.

sequences of deserting the station point. They consist in drawing all horizontal circles as horizontal ellipses, whether opposite the centre of the picture or not ; in always drawing the elliptical representation of the base of a cylinder at right angles with the cylinder itself ; and in drawing all spheres as circles. If a row of columns, moreover, is parallel to the picture, they are always made of the same diameter, as if seen in elevation, and if the direction of the row is slightly inclined to the picture care is taken to diminish their width a little as they recede.

257. Fig. 59, *a*, illustrates these corrections, and shows

Fig. 59, *a*.

further how the same treatment is sometimes extended to the octagon. The right-hand side of the octagonal figure at the top is drawn steeper than it ought to be, not being directed to its proper vanishing point, in order to remedy the apparent distortion seen in the corresponding figure below.

258. Fig. 59, *a*, shows also the effect of applying to vertical circles and semicircles the same corrections as to horizontal ones. The circular window which in the figure below is drawn in true perspective as an oblique

ellipse is here shown as a vertical one, a change which
will probably be regarded by most persons as an im-
provement. The effect of a similar correction in the
semicircular-window head beyond is less happy; it
makes the nearer half look much too big, and ob-
viously throws the imposts, or points where the arch
begins, quite out of level.

259. Since these so-called corrections change and
generally diminish the apparent size of the circles, cyl-
inders, and spheres to which they are applied, the rela-
tion of these objects to other objects is necessarily
changed at the same time. In the first place, more of
the background has to be shown than can really be
seen. In the figure, for example, the openings between
the columns are increased, and objects are seen beyond
which in point of fact would be hidden. This dis-
crepancy is not very important and in general would
hardly be noticed, but the altered relations between
these circular figures and other objects in their imme-
diate neighborhood is a more serious matter. The
square abacus between the shaft and the sphere that
surmounts it looks too big for its place if left without
correction, and looks smaller than its fellows if reduced
as the sphere and cylinder are. So also when an octa-
gon occurs in immediate connection with a circle. If its
shape is adjusted to that of the *corrected* ellipse, its
want of harmony with the rest of the drawing often
becomes painfully apparent. A satisfactory adjustment
may sometimes be effected by a compromise, the ellipse
being made not quite horizontal and the octagon or

square being not quite harmonized with it.   But a per-
fectly satisfactory adjustment is in some of these cases,
and notably in the case of a row of columns, almost
impossible.

These difficulties are of course greater as the objects
in question are further removed from the centre of the
picture, and may be diminished or removed altogether
by so taking the position of the picture and that of
the spectator that the circular object is at or near the
centre C.

260. Although the distortions of circular and spheri-
Distortion of all figures. cal objects are, in general, the only ones that
call loudly for correction, and it is to them
alone that correction is systematically applied, it is ob-
vious that a similar distortion must exist for all objects
equally distant from the centre of the picture, the so-
called distortion consisting in this, that the shape of the
object in the drawing is different from the shape which the
real object presents to the eye (224, 225).   This is in fact
implied in the fundamental principle of perspective, the
principle that a perspective drawing will look right from
only one point, namely, the station-point.   Now as from
the station-point every part of the picture except the
centre is viewed obliquely, *askance*, as it were, every-
thing *must* be drawn of a different shape from what
it appears in order that when the drawing is looked at
thus obliquely it may appear as the object itself does
when looked at directly.   By the very theory of per-
spective only the object just opposite the eye, seen

along the axis of the picture, just at its centre, is drawn
as it looks. Everything else is, so to speak, distorted.
The outline given to it is not its real outline, but one
which will look like its real outline when seen side-
ways from the position assigned to the spectator. This
distortion is inevitable, and every object in a perspec-
tive drawing, except the one at the centre, is always
distorted.

This is most noticeable when objects are drawn in par-
allel perspective, as, for example, in Fig. 23, Plate VI.
But all objects are more or less distorted and exag-
gerated in size. They are *stretched out* in a direction
away from the centre of the picture, just like the
shadows in Fig. 60.

261. The disfigurement produced by this does not
of course become very obvious, except for circles and
spheres, so long as objects are not far removed from the
centre, that is to say, so long as the picture is of moder-
ate extent. The limit commonly assigned to a perspec-
tive drawing is sixty degrees, that is to say, the width
of the picture should not be greater than its distance
from the station point. But this implies that the centre
C is in the middle of the picture, which as we have seen
is often not the case, and it is better to say that no part
of the picture should be distant from the point opposite
the eye more than half the distance of the spectator in
front of it. But even within this range the distortion
even of rectilinear objects is sometimes intolerable, and
great caution must always be used in regard to objects
situated at the edges of the picture.

262. This limit of sixty degrees is obviously an
arbitrary one, and only means that by the
time it is reached the distortion begins to be
noticeable.   It is foolish to say, as is sometimes said,
that this is fixed because sixty degrees embrace all that
one can see without turning his eyes, or as others say,
without turning his head, and that this is accordingly
the natural range for a picture.   For one has to turn his
eyes, more or less, to see directly anything larger than a
pin-head, held at arm's length ; and he has no need to
turn his head even to embrace a horizon of ninety de-
grees.   Besides, why should not one turn his head as
well as his eyes in looking at a picture as well as in
looking at nature ?   If he is at the station-point, where
he ought to be, turning his head cannot make things
look wrong, and if he is not there keeping it still will
not make them look right.

263. There is, nevertheless, a remarkable difference
between turning the head to look at a thing, and
merely turning the eye.   The plane of the picture, so
to speak, regarding the aspect of nature as a picture,
is conceived to be parallel to one's face.   Turning the
head seems to alter the position of the plane of projec-
tion.   If one looks straight down a street he seems to
see it in parallel perspective.   If, keeping his eye fixed
on the same spot, he turns his head, the street seems
now to be in angular perspective, though the image on
the retina has not been disturbed.   The right-angles
will seem to become acute and obtuse.   A horizontal
circle on the floor or ceiling will look like an ellipse

with a horizontal axis if you face it. Turn your head away, without moving your eyes, and though it really looks just as it did before, being regarded from the same point, its axes will seem inclined.

264. What has been said of cylinders and spheres strictly applies to the human figure, which The human may be regarded, in a rough way, as a cylin- figure. der surmounted by a sphere. Perspective distortion is here even more intolerable than in the case of the more exact geometrical solids, and the need of correction is more imperative.

This is excellently illustrated by the phenomena of the familiar parlor amusement called "Chinese Chinese Shadows," in which a sheet is hung across the Shadows. middle of a room and the shadows of the performers on one side are thrown upon it for the entertainment of spectators on the other side. A single lamp is used, and it is obvious that all the shadows except the one just opposite the lamp and on a level with it, must be more or less distorted as they are more or less removed from this centre. But it is also obvious that if one of the spectators places himself exactly opposite the lamp and as far in front of the screen as the light is behind it, the distorted outlines will be foreshortened into the true shape of the figures on the other side as seen from the place occupied by the flame.

265. An historical picture then, if painted in true perspective, with all its figures so drawn as Historical to present their true aspect to the spectator pictures.

standing at a given point in front of it, would have all
its personages as much out of drawing as are Chinese

Fig. 60.
Shadows upon a screen. Fig. 60, which exhib-
its the results of an experiment with a group
of statuary and with half a dozen round balls, illustrates
these conclusions.   They are, as happens in perspective
to all objects (260), *stretched out*, in a direction away
from the centre, C.

266. No such picture of course was ever painted,
painters always adopting the same course for figures
that has been recommended for their geometrical pro-
totypes.   Every figure is outlined independently of all
the others, and in its natural proportions, just as if it
occupied the centre of the picture.   In order to see it
correctly the spectator has to stand opposite to it,
and as he cannot of course stand exactly opposite to
more than one figure at a time, it follows that he can
never see more than one at a time in correct drawing.
All the others are distorted by foreshortening.   But if
he is at a distance from the picture this distortion is
not noticeable, and when he comes near he confines his
attention to the figure nearest his eye.

This is very well illustrated by Raffaelle's fresco

Fig. 63.
The School
of Athens.
called the "School of Athens," of which the
principal part is shown in Fig. 63.   The
figures on the extreme right and the spheres which
they carry are drawn, as has been pointed out by M.
Thiébaut, just as if they were in the centre of the pic-
ture, opposite the eye.   If the spectator were in the
room, one wall of which is occupied by this picture, the

chief one of these figures would appear foreshortened, as in Fig. 64, *a*, the sphere looking like an egg on end. One sometimes in the scenery of a theatre sees a round ball on a post present a similar aspect, from the same cause. In order to make the figure appear as it should, to make it assume when seen from the middle of the room the form intended by the painter, he would have had to draw it as shown in Fig. 64, *b*, giving the sphere a flattened form, just as in Fig. 59 *c*.

Fig. 64, *a*.

Fig. 64, *b*.

267. If his picture is a large one, the painter often has a difficult task to reconcile his background and accessories, which are drawn according to perspective rule, and calculated to be seen from a single point, with his figures, which utterly violate these rules, and permit and indeed require the spectator to regard them from half a dozen different positions. The background proper may not give much trouble. More of it will be seen between the figures than would be the case in nature, as has been pointed out already in regard to columns, and care must be taken not to use any forms which require the spectator to remain exactly at the station point; for this he will not do. But that can easily be managed. The chief difficulty is found in fitting the figures into the foreground. If a chair, for instance, occupies one end of the front of the picture and is put into perspective along with the walls and floor, so as to appear correctly from a point opposite the middle of the picture, it must needs look more or less crooked when looked at from a point opposite the end

The architectural background.

of the picture, and it is no easy matter to make a figure painted from that point of view look as if he were seated comfortably in it. If, moreover, he is to be represented as looking straight across the picture, it is by no means clear whether he should be drawn in profile, as he would appear from this point, or with the three-quarter face which he would show from the other.

268. There is even greater difficulty in reconciling the perspective of a floor with the want of perspective of the feet that stand upon it. If a number of persons are shown in the foreground of a picture all facing the same way, it is impossible to make the direction of their feet agree with that of the boards on which they stand.

Fig. 61, a.

Fig. 61, *a*, shows how the feet of a dramatic company, seen just as the curtain is descending, would be drawn in true perspective, agreeably to the perspective plan below. Fig. 62 shows the neces-

Fig. 62.

sary correction, each pair of feet being drawn just as if it were exactly opposite the spectator. But it is to be noticed that the end man is necessarily represented as standing diagonally across the floor boards.

269. In general, the attitude of the figures at the edge of a large picture is not very clearly defined, and varies as the spectator changes his position. In Guido's "Aurora," for instance, if one stands opposite one end of the picture the figure at the extreme left seems to be marching along the front, and Aurora herself to be looking out of the frame. If he goes to the other end she seems to be looking back at Apollo in the chariot, and

the other figure seems to be just coming round the corner, so to speak, from behind his car. The analogous phenomenon of a portrait seeming to follow one with its eyes is commonly observed. It is not so frequently recognized that the whole face seems to turn, especially when a front face is shown, the cheek, not the nose, seeming to be the most prominent feature.

In the upper part of very high pictures the heads have to be drawn as they appear when seen from beneath, the under side of the jaw and of the eyebrows being shown. But this makes the head seem to fall backwards as one recedes from it. In Titian's "Assumption," for example, the attitude of the head varies greatly according to the position of the spectator. From the opposite end of the room in which it hangs, the face seems to be turned up and the head thrown back. As one approaches the picture it seems to bow forward.

Another difficulty in the perspective of figure-subjects is, that if the figures are as large as life, everything nearer than they are becomes colossal. But this may be avoided, and generally is, by not having anything in particular in front of the principal figures.

270. All the difficulties encountered with the human figure are met in even greater force with figures of animals, except that it is possible for them to be considerably out of drawing without detection. Fig. 65, which is borrowed from the work of M. Thiébaut, illustrates at once the extent of the distortion and the difficulty of correcting it. The form of the

Fig. 65.

pedestal controls the position of the horse's hind legs, and necessitates a distortion for which there seems no remedy. A distortion which is hardly noticed in a four-legged table is intolerable in a quadruped. They should never be drawn in parallel perspective.

CHAPTER XIII.

CYLINDRICAL, CURVILINEAR, OR PANORAMIC PERSPECTIVE.

271. THE previous chapter has discussed the so-called distortions to which circular, cylindrical, and spherical objects are subjected when drawn according to the methods of plane perspective, and has explained the so-called corrections which are applied to such objects. Similar distortions, it was shown, attend the putting of the human figure and animals into perspective, and similar corrections apply. Indeed, it was pointed out (260) that every object not exactly at the centre of the picture must necessarily be more or less out of drawing, though the distortion is not generally such as to attract notice save in the cases mentioned.

272. Plate XV. demonstrates the existence of these distortions, exhibits some instances in which Plate XV. they are intolerable, even in the case of recti- Distortions of rectilinear linear objects, and shows yet another way of objects. correcting them. By *distortion*, as has been said, we mean that the outline given in the drawing is different from the outline presented to the eye by the object drawn. Now the rays of light that pass from the outline of an object to the eye form an irregular cone, whose base is this outline itself (255). The perspective repre-

sentation of this outline is the line in which this cone of
rays is cut by the plane of the picture. If this plane cuts
the cone of rays in a direction at right angles to its axis
— that is to say, if the object is at the centre of the pic-
ture — then the section is of the same shape as the base ;
the perspective is of the same shape as the object.   But
if the plane of the picture cuts the cone of rays obliquely
— as must be the case with all objects not just at the
centre — then the section is not of the same shape as the
base, and the perspective does not look like the object;
it is, so to speak, distorted.   Of course, when seen from
the station point, obliquely, the perspective is *fore-
shortened*, and looks just as the object does.   But in it-
self, and when looked at merely as a line, it presents a
different form (224, 225, 256, 260).

273. This is illustrated in Fig. 70, in which is seen, at *b*,
a rectangular block, drawn in parallel perspec-
tive, but considerably to the right of the centre.

Fig. 70.

Its aspect is such as no rectangular block could ever
possibly present to the eye.   It exhibits three faces,
one of which is a square.   But if a rectangular block is
held so that one of its faces shows four right angles, it
must be held so that neither of the other faces can be
seen at all.   If, on the other hand, it stands so that all
three faces are seen, as this block evidently does, then
all the angles must appear either acute or obtuse.   The
figure within the circle shows how such a block really
looks when one looks straight at it, and this is the way it
is drawn when at the centre of the picture.   The differ-
ence between these two representations exemplifies the

distortion to which all shapes are subjected when the line from the object to the eye is not at right angles to the picture.

But this distortion in the drawing is corrected, by foreshortening, when one looks at the drawing from the station point, S, which in this case is a few inches in front of C, in Fig. 66. In fact, Fig. 70, *a*, was sketched from this point, and is a view, not of the cube itself, but of the Fig. 70, *b*, thus foreshortened into a real likeness of the object it represents.

Fig. 70, *a*.

Fig. 70, *b*.

274. Fig. 66 exhibits other and even more striking phenomena. Take first the church on the left hand. It is horribly out of drawing, although the picture does not extend, on this side, very far from the centre. Not only is the church and the belfry twisted out of shape, but their dimensions increase instead of diminishing as they recede from the eye. This sort of distortion is often seen in old-fashioned prints, and in photographs of very long buildings, or of interiors, taken nearly in elevation. It arises, as is obvious, from both vanishing points, V^R and V^L, being on the same side of the object.

A street.

275. It is accordingly a maxim in perspective that the objects represented must lie between their principal vanishing points, and that the plane of the picture must be so taken as to effect this. In other words, the angular range of the picture to the left of the centre, C, must not exceed the angle between the plane of the picture and the right-hand side of the object, and *vice versâ*

In the figure, for example, the front line of the church and houses makes an angle of about 20° with the picture. V^L is accordingly about 20° to the left of C, and that is as far as the picture of objects whose lines are directed to V^R and V^L, like these, can be extended in that direction.

276. If then it is desired to embrace a considerable extent of horizon, and at the same time to represent objects as being nearly parallel to the picture, there is no choice but to take the plane of the picture in such a way that they shall be *exactly* parallel. That is to say, Parallel Perspective must be employed. For where a rectangular object has one of its sides very nearly parallel to the picture the horizontal lines of the other side must be nearly perpendicular to it, and their vanishing point very near the centre, C. Unless, then, the centre is set quite at the edge of the picture, which is undesirable, the distortion shown in the figure must occur.

If, for example, the opposite sides of a room are both to be shown at once, it will not do to set Interiors. the end of the room at an angle with the picture, however acute; for the end wall will contain the vanishing point of lines parallel to the sides, and one of the side walls that of lines parallel to the end. The other side will then be beyond both vanishing points, and must experience this disagreeable twisting, as is often seen in paintings and photographs. It Streets. is the same with street scenes. If both sides of a street are to be seen at once, they must be perpen-

dicular to the plane of the picture. Fig. 66 sufficiently shows what will happen if they are not.

277. The distortions at the other end of the picture, however, though less offensive, and consequently much more common, are almost as great. The buildings are too long and the hills not steep enough. Both have quite different proportions from what they would present to a spectator at S, and are quite unlike any sketch that would be made of them from that point. For the proportions of an object, that is to say, the relative size of its parts, depend upon the relative angular dimension of the parts, that is, upon the relative size of the angles they subtend at the eye. The apparent distance apart of the points that define them right and left, or up and down, is angular distance. A painter, then, Angular dimensions. who would represent things in their true proportions, as they look, and in their apparent relations one to another, would have to proportion the linear dimensions upon his canvas to the angular dimensions of his object. And this, in fact, is just what every painter, every draughtsman of whatever kind, always does when he undertakes to sketch from nature. It is the method of every artist who undertakes, outdoors or in, to draw things as he sees them; he can have no other; he must give to the representation of objects the apparent shape and the relative size that the objects themselves present to his eye. In other words, he proportions the linear dimension upon his canvas to the angular dimension of the object. But this is exactly what perspective does not do. In sketching, one may

begin in the middle, fix the position of his central ob-
ject, and he will naturally distribute other things about
it to the right and left, according to their apparent dis-
tance from it.   He proportions their distance to their
angular distance, and their size to their angular dimen-
sions ; that is, to the difference of the angular distance
of their edges.   But in a perspective drawing, as is
clearly shown in the plan, the distance of an object
from the centre of the picture is not proportional to its
apparent, or angular, distance, but is constantly greater,
being proportionate to the *tangent* of the angle, and its
size is accordingly proportioned, not to its angular di-
mension, but to the difference of the tangents of the
angular distance of its edges from the centre. The scale
to which they are drawn accordingly increases from the
centre outward, just as in Mercator's Projection, which
gives, indeed, a sort of perspective view of the terrestrial
sphere, as seen from a station point at its centre.

278. In view of these evils, and of the practical im-
possibility of escaping them by using the station point,
the attempt has been made to avoid them by represent-
ing. objects just as they appear, *making the linear di-
mensions in the drawing proportional to the apparent or
angular dimensions in nature.*   It is plain that this
scheme could be thoroughly carried out only by draw-
ing on the inside of a hollow spherical surface, a condi-
tion impossible to fulfil. A cylindrical surface, however,
answers nearly as well, especially when, as is usually the
case, the vertical dimensions are relatively small. A
cylinder, moreover, has the advantage of being a *devel-*

opable surface; it can be rolled out flat. This is the surface employed in circular panoramas, and it is virtually that employed in sketching from nature. For as one turns from one object to another he virtually keeps the corresponding part of his canvas directly in front of him, at right angles to his line of vision, just where a cylindrical surface would be.

279. The plate illustrates the result of this procedure, and affords an opportunity of comparing it with the results of plane perspective. In the plan of the street we have the position of the spectator indicated at S; that of a transparent plane, representing a picture plane, at $p\,C\,p$, and that of a transparent cylinder at $a\,C\,b$. The centre of the perspective picture is at C, the point nearest the spectator, and the plane and cylinder are tangent at that point. Visual rays drawn from the principal points in the street to the station point pierce both surfaces, and pictures drawn upon them would, when seen from the point S, obviously coincide with each other and exactly cover the objects represented. *Projection upon a cylinder, instead of upon a plane. Figs. 66, 67.*

280. Fig. 66 exhibits the result, as shown on the plane $p\,p$, and Fig. 67 that shown on the cylinder $a\,b$. The first strikingly illustrates what has been said of the inevitable distortion of objects in plane perspective, and of their gradual exaggeration of scale as they recede from the centre. Fig. 67 shows the effect of making the linear dimensions in the drawing correspond to the angular dimensions of the objects drawn, that is to say, of drawing everything just as it appears. Of these

effects the most noticeable are these : that in the first place the distortion of the church on the left entirely disappears, and in the second place the distortion on the right disappears also, the houses and the landscape beyond being reduced to dimensions proportioned to the dimensions given to the nearer objects, while the size of the picture is greatly diminished. All this is a great gain. But on the other hand the horizontal parallel lines are all more or less curved. The lines which in

Straight lines drawn as curves.
Fig. 66 are all straight and converge to a single vanishing point, in Fig. 67 converge towards these two vanishing points. Indeed, every horizontal line, except those in the plane passing through the eye, is drawn as curved. Its perspective lies in the line in which the plane passing through the eye and the line itself intersects the cylinder. This is an ellipse in space, and its development is a curved line, concave towards the horizon, which it crosses at points 180° distant from each other, the perspective of its vanishing points. This curvature would of course disappear if the paper were bent into a cylindrical form, and the eye placed at the axis of the cylinder, opposite the horizon, and in the large circular panoramas which are sometimes exhibited, and which have given to this method the name of Panoramic Perspective, this of course is done. But in general the developed cylinder has to remain flat, and it must be confessed that this curvature of lines, which in nature are straight, is itself a distortion which most persons find extremely objectionable.

281. It is worth while to remark, however, that this phenomenon of the apparent curvature of *Straight lines often seem to be curved in nature.* straight lines is of constant occurrence in nature ; and it is just one of those phenomena of nature with which perspective has to do, being concerned with the appearances of parallel lines. All systems of lines which are long enough to indicate both their vanishing points, converging to one point on the right and to another on the left, have an apparent curvature. Such, as has already been pointed out, are the long parallel lines of cloud which often cover the sky, or the sunbeams and shadows which sometimes at sunset pass completely over from west to east. In both these cases each particular cloud or sunbeam, as one looks at it, seems quite straight ; but all the others on either side seem concave towards it. In fact, as they all meet, or tend to meet, at two different points, and to separate between them, they *must* seem curved ; *straight* lines can meet at only one point.

It is the same with the horizon itself, which seems straight when one looks at it, but seems curved when one looks up or down. So with other long lines, such as eaves, sidewalks, or housetops. As one turns his eye rapidly from one end of a street to the other, the apparent curvature reveals itself unmistakably.

Now as all straight lines in nature, if prolonged, *seem* to be curved, approaching the horizon at the gradually increasing angle, so in Panoramic Perspective, their perspective representations *do* curve, until they cut the horizon.

282. To one who is accustomed to observe this curious phenomenon, the curvature of the lines in Cylindrical. or, as we may now call it, *Curvilinear*, Perspective is but a trifling evil, hardly to be counted against its manifold advantages. Of these the chief is perhaps, as has been said, the perfect conformity of its results with those obtained in sketching from nature. Of this an excellent

Fig 68. illustration is afforded by Fig. 68, a rude outline sketch from a water-color by Turner, representing the Ducal Palace at Venice, and the adjacent buildings. He sketched each building just as it looked, and did not mind the resulting curvature of the horizontal lines of his drawing.

This drawing exhibits, however, what is perhaps as objectionable a distortion as any, an apparent convexity in the objects represented. The quay, which in fact is straight, looks convex, as does also the village street in Fig. 67.

In photographs taken with a revolving camera, the

Photographs. pictures are virtually taken upon a cylindrical surface, and exhibit the same phenomena. Photographs taken with a common camera are in Plane Perspective.

283. This convexity is not, however, very noticeable, except in cases like these, where a long line is parallel to the cylinder near the middle of the picture, its perspective being horizontal in the middle and curving to the right and left. Where the subject is so chosen that horizontal lines occur only at the ends of the picture, so that the lines curve only one way, both the curvature

and the convexity are less noticeable; and they can hardly be detected where the lines are short and broken. In such cases the special characteristic of Curvilinear or Panoramic Perspective, that it permits the limits of the picture to be extended indefinitely without the rapidly increasing distortions to which Plane Perspective is liable, can be taken full advantage of. A drawing in Curvilinear Perspective may often be made to embrace a hundred, or even a hundred and twenty degrees of horizon, with less embarrassment than is in Plane Perspective incurred by sixty. Fig. 69, which is borrowed, though much reduced, from a rare and little-known work, by Mr. W. G. Herdman, published in Liverpool in 1853, exhibits this excellence in a striking degree. It represents the meeting of two streets in some foreign town, and succeeds in showing both sides of both streets, without distorting any part of either. The horizontal angle embraced must be more than a hundred degrees. Most of the dotted curved lines, which in the original were carried across the picture in order to show the theory on which the drawing is constructed, are here omitted. Where, as here, the sky-lines are broken, the principal objects in angular, not in parallel perspective, and the continuous horizontal lines few, the disadvantages of this method, as is evident from the figure, are reduced to a minimum.

The extension of the range of the picture.

Fig. 69.

284. Whether in any given case plane or cylindrical perspective is to be preferred is a matter of judgment, and one's decision must depend chiefly upon the nature

of his subject. For architecture, except in picturesque sketches, the latter is in general obviously unfit; but for the landscape painter it affords the same means of escape from the inevitable distortions of plane perspective that the painter of figures finds in the corrections described in the previous chapter.

285. The application of Plane and of Cylindrical Perspective to the same object giving such different results, and Cylindrical Perspective being, Rectifying sketches. as we have seen (277), the method one naturally adopts in sketching from nature, it follows that any attempt to apply the principles of Plane Perspective to such sketches must lead to confusion. This attempt is, however, constantly made: artists, in trying to avail themselves of the materials they have collected in their note-books, frequently resorting to the rules of perspective to correct the inconsistencies and errors, and to fill in the omissions of the originals. The vexation and trouble into which this inevitably brings them — for a drawing made upon a cylinder cannot be treated as if it were made upon a plane — has produced a very general impression among them that the principles of perspective, though true in theory, as they say, are practically false; that they doubtless work very well for geometrical work, like architectural drawings, but that when applied to nature, to the delineation of real things, even to real buildings, they break down.

286. It is indeed plain that such sketches must be interpreted in the light of the system according to which they were made. It is the principles and rules of Cur-

vilinear or Panoramic Perspective that must be called in. This being so, it is worth while to inquire on what geometrical principles this system rests, what are its practical methods, and what are its relations to the system of Plane Perspective.

287. If the picture is supposed to be drawn not upon a vertical plane, but upon a vertical cylinder, Geometrical with the spectator at the centre, or axis, it is principles. plain that, so far as concerns the horizontal dimensions of objects, their perspective representations will be exactly proportionate to their apparent angular dimensions, a given linear measure will correspond to a degree of arc on the horizon, and a given length of horizon in the picture to the whole circumference of 360°. Every part of this horizon will be equally near the station-point, and every point in it may, in turn, be considered as the Centre. We thus entirely avoid those distortions which, in Plane Perspective, necessarily result from the visual rays crossing the plane of the picture at an acute angle, as they must do everywhere except just opposite the station-point. All the rays, at least the rays coming from all points of the horizon, cross the picture at right angles; everything is, so to speak, at the Centre.

And just as in Plane Perspective the perspective of a straight line is the line in which the plane of the picture is intersected by a plane of rays passing through the line and also through the eye, or station-point; and just as the trace of every system of planes is the intersection of the plane of the picture by a plane passing

through the eye parallel to the other planes of the system; so now the perspectives of lines, and the traces of systems of planes, are the lines in which these planes of rays intersect the *cylinder* on which the picture is to be drawn. This, which has already been illustrated in Figs. 66 and 67, Plate XV., is more fully set forth in Fig. 71, Plate XVI. Here S, in plan and elevation, represents the station-point in the axis of the cylinder of the picture. If, now, a, b, c, and d, in the elevation, are parallel horizontal lines, shown in plan at L, the plane of rays lying between them and the eye will cut the cylinder A at a', b', c', and d', the line of intersection being an ellipse, or, at c', a circle. If the cylinder is turned a quarter round, as at B, these lines of intersection will appear as semi-ellipses at b'' and d''; the semi-circle c', however, appears still as a straight line at c'', while the line a', lying at an angle of 45°, appears as a semi-circle at a''.

Plate XVI.
Fig. 71.

In like manner the line e, at right angles to these lines, and shown in plan at R, gives the semi-circle e'' in A and the line e' in B.

288. If now the cylinder is developed, these lines of intersection will appear as in Fig. 72. The circle of the horizon at c will become a straight line; the vanishing-points v^l and v^r, $v^{l'}$ and $v^{r'}$, will appear at V^L and V^R, $V^{L'}$ and $V^{R'}$, 90° apart; while the semi-ellipses a'' b'' d'' e'' will be developed as the curves a''' b''' d''' e''', parallel to the horizon at their highest point, representing the point where a b d and e are nearest the eye, and converging towards

The development of the cylinder.

Fig. 72.

their vanishing-points, as parallel lines ought to do. The height of the lines a''' and e''', representing a and e, which lay 45° above the horizon, is equal to the radius of the cylinder. If the secant planes, $a' b' d'$, Fig. 71, had been carried entirely across the cylinder, forming whole ellipses, as shown by the dotted lines, these curves would be continued beyond the point $V^{L'}$ on the opposite side of the horizon, as far as the point $V^{L''}$, which is, of course, the same point as V^{L}, being 360° distant from it.

289. These curves, always concave towards the horizon, with points of contrary flexure where they cross it, and points of maximum curva- Sine-curves. ture where they are at their greatest distance from it, are what are called in geometry sine-curves, or cosine-curves, and are similar to those obtained by the projection of a regular spiral. It will be observed that the angles at which they cross the horizon are the same as those in Fig. 71, A, and measure the angular distance of the lines represented above the horizon at their nearest point. The maximum distance of the perspective of each line from the horizon is proportional to the tangent of this angle.

290. The perspectives of vertical lines and the traces of vertical planes are of course vertical, and Vertical and these lines are straight, being elements of inclined lines and the cylinder, and they remain straight when traces. the cylinder is developed. The perspectives of inclined lines and the traces of inclined planes are sine-curves, since they too are the lines in which the cylinder is cut

by planes; but their vanishing-points, instead of being upon the horizon, are above and below it, as, for example, V^N and $V^{N'}$ in Fig. 72. But they are all parallel to the picture at the point where they are nearest to it and to the eye, half way between their vanishing-points, and there have their real inclination.

291. It is a property of a *system* of sine-curves, —

Systems of sine-curves. that is to say, a series of curves crossing the horizon at the same point, like the perspectives of a system of parallel lines, — that vertical lines drawn across them are divided proportionally to the maximum heights of the curves, and that for each vertical line the tangents to the curves, at the points of intersection, meet at the same point on the horizon. It is necessary, then, to construct only one line of such a system by developing the line of intersection. The position of any other line, of which the angular distance above the horizon, and consequently the maximum height, is known, may be found at any point by

To construct the sine-curves. drawing a vertical line, and its direction at that point may be found by drawing a tangent. In this respect a number of sine-curves drawn between the same two points are analogous to a number of ellipses with the same major axis (265), as illustrated in Fig. 48, Plate X.

In Fig. 73, for example, the distances cut off by the

Fig. 73. curves upon $h\,h$ are proportional to those cut off upon $t\,t$, as may be seen at $t'\,t'$, and the tangents at the points of intersection all meet at V_p^L. These tangent lines lie in a plane tangent to the cylinder, before

it is rolled out flat, along the line $t\,t$. This relation is shown in Fig. 71, Plan, from which it is clear that these tangent lines are the perspectives upon the plane $p\,p$, regarded as a plane of the picture tangent to the cylinder at the point t, of the same lines whose perspectives are given upon the cylinder by the curves. Of course they meet at a point. This point, V_P^L, is then the vanishing-point in Plane Perspective of the same lines whose vanishing-point is found in Curvilinear Perspective at V_C^L. The plan in Fig. 66, Plate XV., exhibits similar relations.

292. These considerations make it practicable not only to draw all these curves without difficulty when one of them has once been constructed, but in practice to dispense with all but one entirely, treating each separate portion of a panoramic picture as if it were drawn on a vertical plane tangent to the cylinder in that place. Plane Perspective as an auxiliary method. One sine-curve sufficient.

This is illustrated in Fig. 74, a, where we see, as in Fig. 66, an object, L R, the station point at S, and, between them, both a cylinder and a plane of projection. The object not being very large, its picture on the cylinder will not differ perceptibly from its picture upon the plane; and the latter, being easier to draw, may be substituted for the other, using V_P^L for V_C^L, and V_P^R for V_C^R. Fig. 74.

293. Fig. 74, b, shows that, in laying out a panorama, it is not necessary to draw even a single sine-curve, the points V_P^L and V_P^R being ascertained No sine-curve necessary.

directly from V_C^L and V_S^R, in accordance with the following rule : —

Given the Horizon, the vertical line of tangency, and the vanishing-points on the cylinder, set off below the Horizon, on the vertical line of tangency, the radius of the cylinder; with this radius describe an arc tangent to the Horizon; on this arc lay off, right and left, the distances of V_C^R and V_C^L; prolong the radii through the extreme points thus attained, and the points where they strike the Horizon will be the vanishing-points required, V_P^R and V_P^L, on the Horizon of the tangent plane. The rest of the construction can then take place as in Plane Perspective.

To find the auxiliary vanishing-points.

This rule explains itself if we observe that Fig. 74, *b*, is the same as Fig. 74, *a*, with the two parts brought together.

294. Fig. 75, which is taken from the etching of Greenwich Hospital, in Turner's "Liber Studiorum," further illustrates the occurrence of curved lines in sketching from nature, when every object is drawn of just the shape and relative size that it presents to the eye; that is, in proportion to its angular dimensions.

Fig. 75.

295. In a drawing upon a vertical cylinder, although the horizontal dimensions are proportional to the angular dimensions of the objects represented, vertical dimensions are proportional to the tangents, as upon a plane. If it were practicable

The use of a horizontal cylinder.

to draw upon a spherical surface, this would not be so, the linear dimensions in every direction being made proportional to the angular dimensions. The employment of a horizontal cylinder to draw upon, or in, would of course effect this for vertical dimensions. This is practicable for isolated objects, and the result may sometimes be seen in sketches of lofty buildings or towers. The curvature of the lines, and the apparent convexity of the object, making the tower look as if it were strutting or leaning over backwards, are, however, quite as offensive as in the case of horizontal lines. But here, as in the case of the street shown in Fig. 69, these defects disappear when the object is of irregular outline; and this method has the advantage, of course, of giving the actual *aspect* of a tower or spire from a given point, with its foreshortening and altered outline, better than the other. Fig. 76 illustrates this, showing at A a view of the spire of the Central Church in Boston, taken from a photograph, and consequently drawn on the principles of Plane Perspective. At B is a sketch of the same spire, taken at a corresponding distance, drawn upon the surface of a horizontal cylinder with a revolving camera obscura.

Fig. 76.

CHAPTER XIV.

DIVERGENT AND CONVERGENT LINES. — REFLECTIONS.
SHADOWS BY ARTIFICIAL LIGHT.

THE phenomena with which perspective has to do are mainly the phenomena of parallel lines, — lines which seem to meet at an infinitely distant, or vanishing-point; and we have seen that the discussion of such lines prepares the way for the elucidation of the rather complicated phenomena of shadows, since the rays of the sun, and the shadows caused when these rays are interrupted, are also parallel right lines belonging to a single system. But it is worth while to examine also the phenomena of divergent or convergent lines, — of lines that is to say, which actually do meet, or tend to meet, at a point which is within a finite distance. These phenomena so closely simulate the phenomena of parallel lines that we may well inquire just how far the resemblance extends. Moreover, just as the sun and moon cast parallel shadows, every terrestrial source of light, being at a finite distance, casts divergent shadows, and it is a matter of practical importance to find out just how they go, and what becomes of them.

296. Now just as parallel lines *seem* to converge, and look exactly as if they were radiating in every

direction, from some point not very far off, situated upon
the line between the eye and the infinitely dis- Convergent
tant vanishing-point, — so lines that *do* radiate ^lines.^
from a point more or less distant look exactly as if they
were parallel lines going off to infinity.   And as a per-
spective drawing has to do only with the appearances
of things, it represents the two classes of phenomena in
exactly the same way.   All it can do, in either case, is
to show a system of right lines, all meeting,
or tending to meet, at the same point.   The ^The apex.^
point which thus simulates a vanishing-point we will
call the *apex* of the converging lines.

297. This consideration leads at once to the solu-
tion of a question in itself somewhat puzzling.   If the
apex, or point in which the lines meet, is in front of the
spectator, it is easy enough to draw the lines of the con-
vergent system radiating from the perspective of the
apex, like the spokes of a wheel.   But what The apex
behind the
shall we do if the apex is behind us, the lines spectator.
being divergent, passing, so to speak, over our head and
around our shoulders, and, as they recede from us in
front, separating from each other more and more.   Such
are the rays emitted by a candle set behind one's back,
and the shadows cast by it.   What becomes of these
divergent beams ?   Where do they seem to go ?   How
shall their perspectives be drawn — so much of them
as extends beyond the plane of the picture — upon that
plane ?

298. The observation made just now as to the exact
resemblance between the phenomena of divergent or

convergent lines, and the phenomena of parallel lines, affords a hint of the answer to this question. Parallel lines have two vanishing-points, 180° distant one from the other, upon a line passing through the station-point. If the spectator, looking in the direction of the parallel lines of the system, sees one of these vanishing-points directly before him, the other will be directly behind him; and on turning right-about-face he will see that instead. So with the rays of the sun. One vanishing-point is in the sun itself; the other, as we have seen, is in exactly the opposite direction, namely, in the shadow of the spectator's head (170).

But as divergent lines exactly simulate these phenomena, and cannot in fact be distinguished in appearance from parallel lines, except by some extraneous indication, they too will seem to have a second point of convergence, opposite the apex, towards which they seem to tend; and when the spectator turns his back upon the apex, and looks in exactly the opposite direction, he will see the position of this false apex directly before his eyes. So with rays of a candle, or any source of artificial light. If he faces it, the rays converge upon it. If he turns his back upon it, they seem to be directed towards the point exactly opposite the candle. This point, as before, is to be found in the shadow of the spectator's head, in that part of the shadow, namely, that we may call the shadow of his eye. That lines diverging from a point behind the spectator must seem to converge towards a point in front of him is plain; for if we suppose a line

The false apex.

to pass through the eye, in any direction, all lines lying in planes that pass through this line will seem to be directed towards one or the other of its vanishing-points.   But the line drawn through the station-point from the apex of these divergent lines is such a line passing through the eye, and the divergent lines are lines lying in such planes.   They necessarily tend, either way, towards the vanishing-points of the line.   Of these vanishing-points, one will be in the direction of the apex, the other in exactly the opposite direction.   This will be the false apex.

299. Fig. 77, Plate XVII., illustrates these points, and also brings into notice the curious phenomena Plate XVII. that present themselves when the apex, or point Fig. 77. of convergence, is neither behind the spectator nor in front of him, but, so to speak, along-side ; that is to say, is just as far from the plane of the picture as The apex in three posi-he is himself.   Let S be the station-point, $p\,p$ tions. the plane of the picture, and 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10, ten right lines lying in a horizontal plane somewhat below the level of the eye, one end of each touching the line $p\,p$ in plan, and the ground-line, $g\,l$, in the perspective view above.   A simple inspection of the figure shows that the parallel lines 1, 2, and 5 have their vanishing-point at V^L ; 3, 4, 6, 8, and 9, which are perpendicular to the plane of the picture, have their vanishing-point at C ; while 7 and 10 have theirs at V^R. The apex behind the Moreover, the lines 1, 3, and 7, which con- spectator verge upon A_1, behind the station-point, and which accordingly appear to the spectator at S to be diver-

gent, have their perspectives directed towards A_1', a false apex, a point in space exactly opposite the apex itself, and as far above the horizon as the apex A_1, on the ground, is below it.

The lines 5, 6, and 7, which converge upon A_2, the apex in front of the spectator, are shown in perspective meeting at the perspective of A_2.

The apex in front.

300. Finally, the lines 5, 8, and 10, which have their apex at A_3, a point just as far from the plane of the picture as the station-point, have their perspectives *parallel*. The same is true of the lines 2, 4, and 7, which converge upon the horizontal projection of the station-point, at S. But these last lines are not only parallel, but *vertical*.

The apex along-side.

301. It appears, then, that if converging lines have their apex, or point of meeting, in front of the spectator, their perspectives will converge upon the perspective of their apex; if it is behind the spectator, they will converge upon a point which we have called their false apex, situated exactly opposite the apex itself, and which is the vanishing-point of a line drawn through the apex and the station-point; if the apex is neither in front of the spectator nor behind him, but just as far from the picture as is the station-point, the perspectives of the convergent lines will be parallel. Their position depends upon the height at which the apex lies, and is most easily determined by finding the perspective of that element of the convergent system that is perpendicular to the plane of the picture (such as line 8 in the figure), and making the rest parallel to that. Finally,

if the apex is directly above or below the station-point, the perspectives of the convergent lines will be vertical. If it is on a level with his eye, they will be horizontal.

302. Fig. 78, a sketch, somewhat reduced in scale, from an old English print, illustrates this last point. It shows three streets converging upon the point occupied by the spectator. The axes of these streets are all drawn vertical and parallel.

Fig. 78.

A similar result would follow if one were to draw the spokes of a large wheel while standing upon the hub. They would all be vertical and parallel. So with the reflections of distant lamps which at night often seem to cross the water and converge below the feet of a spectator standing upon a bridge. In perspec-tive they also would all be drawn vertical and parallel, continuing, indeed, the vertical line of the lamp-posts from which they emanate, as is seen in Fig. 79.

Reflections.

Fig. 79.

303. Fig. 80 shows that the position of the false apex may most conveniently be determined by finding the perspectives of two horizontal elements of the converging series, a and b, one perpendicular to the plane of the picture, and one passing beneath (or above) the station-point. Let a' and b' (or a'' and b'') represent the points in which these lines pierce the plane of the picture, their distance from the horizon depending upon the position of the apex, A, below or above the eye, and the other element of their position being obtained from the orthographic plan.

To find the false apex by perspective.

Fig. 80.

The perspective of the line a will begin at a' (or a''), and be directed towards its vanishing-point, C ; that of the line b will begin at b' (or b''), and be vertical, since it passes directly below (or above) the station-point (301). The point A′ (or A″), where these perspectives would meet, if prolonged, is the position of the false apex.

The same result may be obtained by the methods of Another way, Descriptive Geometry, as may be seen in the by descriptive geometry. same figure. The false apex is the vanishing-point, as has just been seen (298), of that element of the converging system that passes through the eye. But the vanishing-point of a line passing through the eye, and which is consequently seen endwise, is the point where it pierces the plane of the picture. Now b, in Fig. 80, is the horizontal projection of such a line, and a' A′ (or a'' A″) its vertical projection. A′ then (or A″) is the point where it pierces the picture, its vanishing-point, and the false apex in question.

304. Fig. 77 furnishes other illustrations of the false apex besides those already mentioned. The Fig. 77. points of convergence, A_5, A_6, and A_7, situated behind the spectator, have each a false apex, towards which the perspectives of the converging lines are directed, as may be seen at A_5', A_6', and A_7'.

A_5, A_6, and A_7 are supposed to lie in the same plane as do A_1, A_2, A_3, and A_4, previously discussed. It will be observed that the most distant of them, A_6 and A_7, have the apex nearest the horizon, the nearer ones, A_1 and A_5, farther off, and those on a line with the spec-

tator, A_3 and A_4, at an infinite distance, the perspectives of the converging lines being parallel.

If either apex were in a different horizontal plane, the vertical position of the false apex would change accordingly, that of A^7, for instance, appearing at A_7'', if the point of convergence were as far above the spectator's level as it has been supposed to be below it.

305. In determining the form of shadows cast by artificial light we follow the same line of argu- Shadows by ment as when, in Chapter IX., we discussed light. the shadows cast by the sun. Here, as there, the (invisible) shadow of a point is a line drawn through the point from the source of light, and its (visible) shadow on any surface is the point where this line pierces the surface. Here, as there, the (invisible) shadow of a line is a plane in space, and its (visible) shadow upon any plane that receives it is the line of intersection of these two planes, its vanishing point being at the intersection of their traces. Here, as there, the trace of the plane of shadow is found by drawing a line through the vanishing points of two of its elements, one being the vanishing point of the line that casts the shadow, and the other the vanishing point of the (invisible) shadow of some point in that line.

306. The only difference in the two cases is that in the case of sunlight the vanishing point of this last line is always known beforehand. The shadows of all the points are parallel lines, and have a common vanishing point, either in the sun, or in the shadow of the specta-

tor's head, opposite the sun.　But with artificial light
the lines drawn from the source of light to the different
points of the line whose shadow is to be found are not
parallel but divergent.　Each has a direction of its own
and a vanishing point of its own.　Still, all these di-
vergent lines lie in the same plane, the plane of shadow
in question, like the sticks of a fan, and any one of them
will suffice, with the line itself, to determine the trace
of this plane.　For this it is necessary only to select some
one of these divergent lines and determine its vanishing
point.

307.　This does not of course lie in the apex of

To find the
vanishing
point of a ray
of artificial
light.

these converging rays, the source of light itself.
It must be determined just as the vanishing
point of any finite line is determined, when

Fig. 81, *a.* the line is given by two points at its extremi-
ties.　In Fig. 81, *a,* for instance, let M be the line
whose shadow is to be cast, determined in position by
the vertical lines at its extremities, which show its rela-
tion to the horizontal plane.　Let A be the apex of rays,
or source of light, and let a single ray, S', be drawn
through this point and some point upon the given line,
say its lowest point.　These two lines suffice to deter-
mine the plane of shadow, and the trace of that plane,
T S M, will pass through their vanishing points, $V^{S'}$
and V^M.

308.　Fig. 81, *b,* shows how these may be determined.

Fig 81, *b.*

The horizontal line R, connecting the perpen-
diculars let fall from the ends of the line M,
being prolonged until it touches the horizon, finds its

vanishing-point at V^R (for a line lying in a plane always has its vanishing-point in the trace of that plane); the trace of the vertical plane that contains M and R passes vertically through V^R (for the trace of a plane passes through the vanishing-points of every line that lies in it); and the line M prolonged finds its vanishing-point, V^M, on this trace (the trace of the vertical plane in which it lies).

To find the trace of the plane of shadows.

In the same manner $V^{S'}$ is found by drawing, as far as the horizon, a line beneath the ray of light, and by producing that ray, until it meets the trace of the vertical plane in which it lies. T S M, the trace of the shadow of M, is then found, as usual, by drawing a line between $V^{S'}$ and V^M.

The visible shadow of the line M upon the plane, S M on R L, lies at the intersection of the plane of invisible shadow with the ground plane. Its vanishing-point, $V^{SM, RL}$, is the intersection of their traces; that is to say, of the trace T S M and the Horizon. The visible shadow will begin where the ray S', passing through the end of the line M, reaches the ground, and will be directed towards, or away from, this vanishing-point, as in the figure.

A similar result would have been reached, as appears in the figure, by taking rays of light that pass through other points of the line M, such as S'' or S'''. These have different directions, but lie in the same plane; this gives us new vanishing-points, $V^{S''}$ and $V^{S'''}$, but the same trace, T S M, just as before.

309. Any other ray would answer as well, of course, as these; the choice of one or another is purely a matter of convenience. Now just as we found, in studying shadows cast by the sun, that they were most easily determined when the rays of light were parallel with the picture (the sun being neither behind the spectator nor in front of him, but on one side), so here we shall find that there is a practical advantage in selecting for use neither of the rays just now employed, but, instead, the ray that is parallel to the picture, the ray that passes through the line at a point which is just as far from the picture plane as is the apex, A, or source of light.

A second way; the ray parallel to the picture.

Fig. 81, c, shows how this is done. A line parallel to the horizon is drawn through the foot of the vertical let fall from the apex A until it intersects the line R, at which point another vertical is erected to intercept the line M. The line S, from A to this point, is obviously a ray of light parallel to the picture, and the line drawn through V^M, parallel to it, is the trace of the plane of shadows, T S M.

Fig. 81, c.

This is exactly the method employed in Fig. 36, Plate VIII., founded on the proposition that the element of a plane parallel to the picture has its perspective parallel to the trace of the plane (38).

310. The relations between the treatment of diverging light and that of parallel light may be illustrated by these figures. If the source of light, Fig. 81 d, be supposed to retreat to an infinite

Fig. 81, d.

distance, the vertical line dropped upon the horizontal plane will shorten until it rests upon the horizon. $V^{s\prime}$, $V^{s\prime\prime}$, and $V^{s\prime\prime\prime}$, will coalesce with each other and with A, which will now become V^s, and the figures will both assume the aspect of Fig. 81, d.

311. In casting shadows by artificial light, however it is not so important to obtain the vanishing-point of the line of visible shadow as it is in sunlight. For, in sunlight, the vanishing-point of the shadow of a line, when once obtained, serves for the shadows of all the other lines of the system, since the shadows are parallel as well as the lines themselves, and all have the same vanishing-point. But in artificial light the shadows of parallel lines are not parallel, and the vanishing-point of one is of no service in drawing the next one.

It is accordingly just as well, and it is much easier, to find the shadow cast upon a plane by a line exposed to artificial light by finding the shad- Fig. 81, e. ows of its extreme points and drawing a staight line between them. Fig. 81 e shows how this may be done. The point in which the ray S′ touches the ground is one point of the line of shadow, the ray S″ gives a second point, and the point where the line itself pierces the ground gives a third. Either two of these suffice to fix the line of shadow in position and direction.

Fig. 81 f shows that in the sunlight, also, Shadows by Sunlight. we can in like manner determine a shadow by Fig. 81, f. points, without using the vanishing-point.

This is, whether in sunlight or artificial light, the most convenient way of determining the shadow of a

curved or irregular line. It amounts, virtually, as the
figure shows, to letting fall a perpendicular
from each point upon the plane in question,
and drawing its shadow, and then passing a line of
shadow through the extreme points thus obtained, as is
done in Fig. 81 *g*.

Fig. 81, *g*.

312. But a plane given by a line and a point can
be determined by a line drawn through the
point parallel to the given line, as well as by
one crossing it. The same result as in the
previous figures may accordingly be obtained by em-
ploying the method shown in Fig. 81, *h*.

Fig. 81, *h*.
A third way ;
the ray par-
allel to the
given line.

The point V^M having been determined, a ray, S^M, par-
allel to M, is drawn through the apex A, touching the
ground at its point of intersection with the horizontal
line beneath it, the line M being also prolonged to a
similar point. This ray, the line M, and the horizontal
line, L, joining their extreme lower points, are obviously
all in the plane of shadow.

The line prolonged in the other direction, beyond M,
being the line in which the shadow of M intersects the
horizontal plane, is obviously the visible shadow of M
on that plane, the exact length of which can be cut off
by drawing divergent rays from A through the extremi-
ties of M, as in the figure.

The shadow of any line cast on any plane surface by
an artificial source of light may accordingly be found
by drawing a single ray parallel to the line in question,
and finding the points in which both the line in question

and the ray will pierce the plane on which the shadow
falls. A line joining these points will be the
line in which the plane of rays passing through
the line intersects this plane, and, if produced beyond
the point where the line pierces the plane, will give the
line of its shadow.

A general method.

This method seems less direct than that of Fig. 81 *e*,
which is indeed often preferable for single lines. But
the method just explained is to be preferred where sev-
eral parallel lines are subjected to artificial light. For
the point *x*, Fig. 81 *h*, where the parallel ray pierces
the plane upon which the shadow falls, holds the same
relation to one line of the system as to another, the
shadows of all of them passing through it. The shadow
of the line M_2, parallel to M, in the figure, is directed
towards this point. It is then an apex of converging
lines, and is as important and serviceable a point when
the shadows of parallel lines are cast by artificial light,
as their vanishing-point is when they are cast by sun-
light, and answers the same purpose in saving labor.

313. It appears then that, in artificial light, the
shadows upon a given plane of a given system of par-
allel lines all converge to the point in the plane at
which it is pierced by an element of the system passed
through the source of light.

314. Fig. 81 *i*, shows that the method of
Fig. 81 *h*, may also be used for finding shadows
by sunlight, the shadow of a line in a plane
being determined by means of a second line parallel
with it, instead of by a ray of light crossing it.

Fig. 81, i.
This method applied to sunlight.

We cannot, to be sure, draw this second line through the source of light, that being infinitely distant. But it is not essential, even with artificial light, that the parallel line be drawn through the apex of the divergent rays. It may be drawn through any point of any ray. In Fig. 81, *j*, for instance, a point being taken on a ray that passes from A to the upper end of M, its height above the plane is easily determined, and a line parallel to M can as serviceably be taken through that point as through A itself.

But this process can be exactly as well employed with the parallel rays of the sun, as appears in Fig. 81, *i*, where the shadow of M on the ground is again found without the aid of the trace T S M.

315. Fig. 82, Plate XVIII., illustrates the ease with which can be found the shadows of vertical lines upon a horizontal plane. The shadow of each such line is obviously a vertical plane, of which the vertical line

Plate XVIII.   dropped from the source of light, A, is also
The shadow   an element. The line joining the lower ends
of vertical
lines.   of these vertical lines is accordingly the in-
Fig. 82.   tersection of the plane of shadow with the
ground, and its prolongation gives the line of shadow. This has indeed been done incidentally, as was said, in Figs. 81, *e* and *g*.

The figure shows, also, that a similar procedure may be followed in the case of a horizontal line, normal to a vertical plane, the projection of the apex of rays upon that plane being known. (313.)

316. The principle of Fig. 81 h is illustrated in Fig. 83, where the shadows of several parallel lines Shadows of parallel lines. are thrown upon a broken surface composed of several planes. The shadow of each line is Fig. 83. directed, in each plane, towards, or away from, the point where a ray drawn parallel to it through the apex of rays, or source of light, pierces that plane. As the same ray is parallel, of course, to all the lines of the system, one ray suffices for them all, and their shadows, in each plane, are directed towards the same point.

The vertical and inclined lines that cast the shadow lie in a vertical plane, and the vertical and inclined rays taken parallel to them lie in a vertical plane parallel to it. The points 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, in which these rays pierce the planes on which the shadows fall, lie in the line in which the plane of rays cuts those planes.

Of these points, 1, 2, and 3 are those in which the vertical ray pierces the ground and the two inclined planes at the bottom of the wall, and the shadows of the vertical bars cast upon these planes are directed to these points; 4, 5, and 6 are the points at which the horizontal ray pierces the three vertical planes of the wall and base, and the shadows of the horizontal bars cast upon these planes are directed towards these points; 7, 8, and 9 are the points at which the inclined ray pierces these three vertical planes, and 10 and 11 those at which it pierces the two inclined planes, and the shadows of the inclined bars falling upon these five surfaces are directed necessarily towards these five points or apexes.

The shadow upon the vertical wall of the irregular line of the arch and the moulding that supports it, is found by the method shown in Fig. 81, *g*. (311.) Horizontal lines are drawn from successive points perpendicular to the surface of the wall, and a line of shadow is drawn through the shadow of the successive points.

317. It is to be noticed that if one were to stand under the arch, with his eye at the point occupied by the lamp, he would see just so much of the wall as is now in light. These processes enable us then to determine how much of what is shown in a picture is visible to the personages represented in it. In Fig. 83, for example, if we suppose the eyes of the young man standing outside the arch to be an apex of visual rays, and then find the shadow, so to speak, cast by them upon the wall, the point 12 being the projection of this apex upon the plane of the wall, we find that the light of his eyes just reaches the young woman sitting at the window within.

The most frequent examples of shadows cast by artificial light are presented in pictures of interiors lighted by candles or gaslight. It is customary to restrict the discussion of these phenomena to cases in which the source of light is, as in the figures already given, in front of the spectator. But the theory of false apexes explained in this chapter (301) enables us to illustrate also the case, equally common in actual experience, in which the source of light is behind the spectator. Figs.

85 and 86 illustrate these two cases. The shadows are cast by the method of Fig. 81, f and g.

318. Figs. 84 and 85, Plate XIX., represent the plan of a long room, and the perspective of one end of it. The line pp, in Fig. 84, shows the position of the plane of the picture, S is the horizontal prospective of the station-point, and A and B the projection of two gas-burners suspended from the ceiling. The burner, A, appears in its proper position in Fig. 85; its projection upon the floor is shown at A^H, upon the ceiling at A^Z, on the right and left hand walls, at A^R and A^L, and upon the rear wall, which is parallel with the plane of the picture, at A^P. These points are the points in each surface towards, or away from, which the shadows of lines normal to these surfaces are directed when this burner is lighted (314). Each is an apex of the diverging visible shadows, while A is the apex of the diverging lines of invisible shadow. The extreme point of the visible shadow of the normal line is in each case determined by the line of the invisible shadow of the extreme point of that line.

In Fig. 86 the other burner, B, is supposed to be lighted; and we have the case already illus- trated in Figs. 77 and 80, in which the apex of diverging lines is behind the spectator. Following the method of Fig. 80, we find that lines drawn normal to the plane of the picture from these several apexes pierce the picture at the points B_N, B_N^H, B_N^Z, B_N^R, and B_N^L, and the false apexes B', $B^{H'}$, $H^{Z'}$, $B^{R'}$, and $B^{L'}$ are then easily found.

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B′ is determined by the aid of the plan, Fig. 84, the

The false
apexes. line $B_N C$ being the perspective of the ray normal to the picture, and the vertical line through b the perspective of the horizontal ray passing above the spectator. Their point of intersection, B′, is the false apex (303). It lies in the shadow of the spectator's head, just at the point occupied by his right eye, the eye with which he is supposed to be looking. This point is, then, in artificial light as well as in sunlight, the focus of rays of light proceeding from behind the spectator. Lines drawn in like manner from the points B_N^H, B_N^Z, etc., through the centre, C, will contain the other false apexes, and as the true apexes are either on a level with B or are directly above or below it, so the false apexes will be either on a level with B′ or directly below or above it; that is to say, they will lie in the intersection of the lines drawn through C, with horizontal or vertical lines drawn through B′.

319. It will be noticed that the apex, B^P, towards which the shadows of the lines normal to the rear wall are directed, occupies the same position as the point A^P, towards which the shadows of the same lines were directed in Fig. 85. The only difference in the two cases is that the shadows in Fig. 86 are shorter than those in Fig. 85, being cut off by lines converging to B′ instead of by lines converging to A; but they coincide with them as far as they go. A^P and A being near together, the lines directed towards them intersect a good way off, giving longer shadows than do the lines directed towards B^P and B′.

Indeed, there is much less of shadow altogether in Fig. 86 than in Fig. 85, as was to be expected, the spectator's eye being so nearly in a line with the source of light. If it could coincide with it, as was supposed to happen in the case of the serenader in Fig. 83, he would see no shadows at all.

The shadow of the lantern, A, thrown upon the rear wall at A^p, or B^p, is really, of course, a little larger than the lantern, being at a greater distance from the apex, B. But its perspective is smaller than the perspective of the lantern, its distance from the station-point being relatively greater still. If the station-point were exactly on a line between the point B and the lantern, the outline of the shadow would obviously fall within that of the lantern itself.

320. The shadows of convergent lines, whether cast by natural or by artificial light, converge, of course, Shadows of to the shadow of their apex. But if the apex convergent lines by artifical light. is just as far from the plane on which the shadows fall as is the source of light, its shadows will be at an infinite distance, and the shadows of the converging lines will be parallel to one another and to the shadow of that element of the convergent system of lines which is normal to the plane, and will have the same vanishing-point in the trace of that plane; if it is farther off, they will diverge, being directed towards a point in the plane beyond the source of light. This point is, so to speak, the *negative* shadow of the apex, being still the point where the line through the apex of the lines and the apex of the rays pierces the plane. It is the

same point that would be obtained if the two apexes should change places.

321. Fig. 87 illustrates these points, showing the shadows of several geometrical figures cast upon the floor. The point A^H is the horizontal projection of the gas-light, A. The shadows of the lines which meet at the apex B, meet at its shadow, B'. The shadows of the vertical axes are directed upon A^H. The apexes C and C, are just on a level with A; their shadows upon the table are accordingly at an infinite distance, and the shadows of the lines that meet at these apexes are accordingly parallel in space, and have their vanishing-points upon the Horizon. Each point D is further from the table than A, and the shadows of the edges of the pyramid of which it is the apex converge towards the corresponding point D', the "negative shadow" of D, being the point upon the horizontal plane where the shadow of A would fall if D were a source of light.

Fig. 87.

In sunlight also the shadows of converging lines are parallel when the source of light is just as far as the apex from the plane on which the shadow is cast. Fig. 88 shows how the Pyramids at sunrise throw parallel lines of shadow towards the western horizon.

Fig. 88.

The analogy of these phenomena of shadow to those of the perspective representation of converging lines (301, Fig. 77) recalls the analogy, already pointed out, between the shadows of all objects when cast upon a plane by artificial light, and their perspective representation (265, Fig. 60).

CHAPTER XV.

OTHER SYSTEMS AND METHODS.

IN the processes hitherto described every line has been regarded as a portion of an infinitely long line tending towards its vanishing-point, and every surface as a portion of an infinite plane extending to its trace, or horizon; and it is by determining the position of these vanishing-points and traces that the position of the perspective representations of these lines and surfaces has been fixed. This way of looking at the subject involves a comprehensive survey of the phenomena in question, and leads to a proper understanding of their relations. The processes deduced from this study are also generally convenient in practice; for, though some of the vanishing-points are generally somewhat remote, still the space required for drawings executed upon the small scale commonly employed is not greater than can usually be afforded.

Before dismissing the subject, however, it is proper to consider some other methods of obtaining the same results, based upon the consideration of these same phenomena, and involving a more extended application of some of the principles already considered, — methods which under certain conditions offer considerable advantages.

322. Several of these special methods are illustrated
in Plate XX.   In all of them the consider-
ation of vanishing-points and traces is more or
less dispensed with, the lines to be represented
being considered merely as finite lines lying between
two points, the immediate object of the processes em-
ployed being to fix the perspective of these points.   In
some of these methods the abandonment of the outlying
vanishing-points leads to so great a reduction of the
space required for making the drawing that the work is
performed almost entirely within the limits of the pic-
ture itself.   In executing large works, such as scene
paintings or mural decorations, this is, obviously, of
great convenience.

Plate XX. Vanishing-points dis-used; space economized.

The Method of Direct Projection.

323. In this method no use at all is made of vanish-
ing-points, and no reference is had to any of
the phenomena of parallel lines that are rep-
resented by means of them.

The method of direct con-ical projec-tion.

The object to be represented is carefully drawn, both in
plan and in side elevation, and the plane of the picture,
seen *edgewise* or in section, and the station-point are
shown.   By drawing lines, representing the visual rays,
from every point in the object to the station-point, first
on the plan and then in the elevation, and noting their
intersection with the plane of the picture, the horizontal
and vertical position of the perspective of every point
may be ascertained, and a representation of the object
obtained by drawing lines connecting the points.

Fig. 89 illustrates this method, giving at a, b, and c, the plan of a cross, set obliquely, and two ele-

vations, both of which are necessary, as neither one of them exhibits all the points visible from the station-point, S, in front. Lines representing the visual rays are drawn, both in plan and in both elevations, from all the visible points to the point S, and the points where they pierce the plane of the picture, $p\,p$, indicated. These points, being transferred to the side and bottom lines of the figure, 89, d, suffice to determine the position of each point in perspective.

This kind of projection, in which the lines of projection converge to a point, instead of being parallel as in plans and elevations, is called Conical Projection, as distinguished from Orthographic.

The Mixed, or Common Method.

324. The method of Direct Projection is seldom used to determine vertical dimensions, — that is to say, to fix the position of horizontal lines, —

the labor of constructing two oblique elevations being intolerable; but it is very generally employed for the determination of horizontal dimensions, — that is, to fix the position of vertical lines, the length of vertical lines being determined by means of lines of vertical measures and vanishing-points on the Horizon.

Fig. 90 illustrates the application of this mixed method to the subject of the previous figure. The vertical lines are drawn as in Fig. 89, d,

their position being taken from the geometrical plan

at a, by direct projection. Their length is determined by setting off the real heights, as given by the elevation along-side, on a line of vertical measures, $v\,v$, taken where the plane of the front of the cross intersects the plane of the picture. This is fixed by the point m, in Fig. 89, a. Fig. 89 also serves to determine the vanishing-points V^R and V^L, and the corresponding points of distance, D^R and D^L.

325. Though this method is deficient in scientific

The advantages of the orthographic plan.

unity, an entirely different principle being employed for horizontal dimensions from that used to determine vertical dimensions, it is often very convenient in practice, especially when, as is frequently the case with buildings, a carefully drawn ground-plan, prepared for other purposes, can be taken advantage of. This is still the process most commonly employed by architectural draughtsmen for the determination at least of the main lines of their work. Points of distance, points of measures, and the vanishing-points of inclined lines, are employed, if employed at all, only as auxiliaries and alternative devices.

326. But the employment of the Perspective Plan to

The advantages of the perspective plan.

determine horizontal distances, and thus fix the position of the vertical lines of a perspective drawing, as has been done in the previous chapters, is altogether preferred by the best and most recent writers, and by the best informed draughtsmen. It has the signal advantage of avoiding the confusion and error that necessarily attend the

multiplication of points of intersection distributed along
a single line. Even in the figure just given, simple as
it is, we find in 89, *a*, a dozen points crowded together
upon the line *p p*. It is not easy, in transferring them
to Fig. 90, to keep clearly in mind which is which, —
which indicates a point at the bottom, which a point
on the top, which belong to the front plane and which
to the back. In the perspective plan, on the contrary,
Fig. 91, every point is significant ; there is no
confusion, and, the relations of all the parts
being clearly exhibited, there is much less danger of

Fig. 91.

trivial inaccuracies than in a blind and merely mechani-
cal procedure. Moreover, if the perspective plan itself
becomes too crowded with details, it is practicable to
make a second or a third, as has already been done in
Plate III. In the case of high buildings it is usual
to make a separate perspective plan for each story, those
of the upper stories being drawn above them, as those
of the lower stories are drawn below. These plans are
always perfectly intelligible and serviceable after any
lapse of time, and, as has been said, may conveniently
be made on separate strips of paper, thus saving the
drawing itself from disfigurement, and, indeed, protect-
ing it from injury. These strips of paper with the plans
upon them can then be preserved, and in case a second
drawing for any reason has to be made, half the labor of
making it will have been saved.

327. Other and incidental advantages of this method
are the great facilities it offers for designing in perspec-
tive, for working up a perspective drawing from rough

sketches, and altering and adding to it at will, studying the effect of such changes as may be suggested by taste or convenience.   It is also to be observed that the perspective plan takes up less room than the orthographic plan, with its system of visual rays directed towards the station-point, and this is sometimes a consideration of some importance.

328. The reason why the perspective plan is so little used, although the theory of points of distance on which it is based is perfectly familiar, is that unless this plan is sunk considerably below the picture the desired points are not very accurately ascertained, the lines whose intersections determine them cutting each other at an acute angle. Sinking the plan, however, as is done in Fig. 90 and elsewhere, entirely obviates this, and has the advantage not only of enabling one to draw it on a separate paper and preserve it for future use, as has just been suggested, but of keeping the picture itself free from construction lines.

Sinking the perspective plan.

The Method of Co-ordinates.

329. The method of Co-ordinates applies the principles of Parallel Perspective, as set forth in Chapter VII., to the solution of every class of problems.   Lines parallel and perpendicular to the picture are treated as is usual in that system.   Lines inclined to the picture are determined, as in the method of Direct Projection, by ascertaining the perspective of the points between which they lie, their vanishing-points

The method of three rectangular co-ordinates.

being neglected. The position of a point in space being known, the three dimensions that determine its position can easily be put into perspective, two of them being taken parallel to the picture, and the third perpendicular to it; and, the perspective of every point being thus ascertained, the lines lying between them are easily drawn.

In speaking of these three directions, at right angles to each other, it is convenient, just as we call Height, the vertical dimension Height, to speak of the width, and depth. horizontal dimension parallel to the picture as Width, or Breadth, and of the other horizontal dimension, perpendicular to the picture and parallel to the Axis, as Depth.

330. Fig. 92 exhibits the application of this method to the same subject as that by which the other Fig. 92. methods just mentioned were illustrated. The a and b. eye being supposed to be about two inches from the paper, the point of distance would be two inches from C, the centre of the picture. The point of half-distance is accordingly set one inch off, at D½, and the *perpendicular* dimensions are laid off upon the ground-line of the perspective plan in Fig. 92, *b*, at half the scale of the orthographic plan above (Fig. 92, *a*), from which they are taken. (142.)

In Fig. 92, *c*, the vertical dimensions, as given by the elevation in Fig. 90, are laid off upon Vertical dimensions. the scale of heights erected at *g*. Horizontal Fig. 92, *c*. lines drawn from the points thus ascertained to the centre, C, and vertical lines drawn from the points pre-

viously ascertained upon the scale of depths, drawn from g to C, in the plan below, determine by their intersection the height above the ground-plane and the distance behind the plane of the picture of every point in the object to be represented. This enables one, if he pleases to do so, to construct a perspective of the side elevation, as is done in the figure, just as the perspective of the plan is constructed. In fact, Fig. 92, c, is the perspective of Fig. 89, b, just as Fig. 92, a, is the perspective of Fig. 89, a. The perspective plan and elevation being both given, the perspective of the object is easily constructed.

The side elevation in perspective.

It is sometimes convenient to construct this perspective elevation in a vertical plane not perpendicular to the picture; a plane, that is, whose horizontal elements are directed to some other point of the horizon than the centre, C. This is shown in Fig. 92, d. In this case points upon a new line of depths are taken across from the line g C.

Fig. 92, d.

331. All this, though simple in theory, is laborious in practice, as the application of general methods to special problems is apt to be. In most cases it is not worth while to give up the facility and accuracy afforded by the use of vanishing-points for this tiresome and roundabout process; but when the object to be drawn is irregular in shape, or bounded by curved lines, so that it has to be put in by points at any rate, the method of rectangular co-ordinates, according to parallel perspective, best meets the case. Even when such objects occur in a drawing made in angular perspective

it is often convenient to employ it. When, finally, the scale of the drawing is so large, or, what comes to the same thing, the space to work in is so small, that the vanishing-points, are inaccessible, this method is of great service. By employing points of half distance, or quarter distance, etc., the necessary constructions can generally be confined within the limits of the picture itself.

332. The most common application of the principle of co-ordinates is to the determination of the *The scales of height, width, and depth.* size of miscellaneous objects, such as trees, animals, or human figures in landscapes. A vertical scale being established in the plane of the picture, resting upon the ground-line, lines converging to any convenient point on the Horizon suffice to show how large any object, a human figure, for instance, should be drawn in any part of the picture.

This use of a scale of heights is illustrated in Fig. 93. The figures are supposed to be all of the same *Fig. 93.* height as the one in the immediate foreground. *Figures, etc., in a land-* The scale of heights, on the left, shows how *scape.* tall such a figure will appear at every point of the horizontal plane between the ground-line and the Horizon. The position of such a figure above or below that plane will not of course affect its apparent size. The man in the balcony, on the right, for instance, is drawn just as tall as the man on the platform beneath, and the persons upon the inclined plane descending to the water are of the same height as those upon the pavement alongside.

The size of the different vessels is determined in a
similar way.

333. It is worth while here to point out that though
points of half-distance, quarter-distance, etc.,
in Parallel Perspective, do not serve, as do
points of distance, as vanishing-points of lines
of 45°, such lines can nevertheless easily be
drawn through any point by their aid.

Let a and b in Fig. 94 be two points through which
it is desired to draw lines making 45° with the axis
and with the ground-line, the centre, C, and the point
of half-distance, $D\frac{1}{2}$, being given. By drawing through
these points lines directed towards C and $D\frac{1}{2}$, crossing
them with a line parallel to the horizon, and then
doubling upon this line the distance intercepted, lines
may be drawn which are obviously directed towards
$D\text{-}V_1^x$.

If the point of one third-distance is given, the inter-
cepted portion must be trebled, as at c, or quadrupled,
as at d, if the point of quarter-distance is used.

It is hardly necessary to explain how a square can be
erected on a given line parallel to the ground-
line, as is shown in Fig. 94, using points of
half, third, and quarter distance.

The Method of Squares.

334. The processes of the method of co-ordinates may
be much simplified, especially in the case of
objects irregular in plan, by adopting the de-

vice of *squaring*, commonly used by draughtsmen to assist them in copying the outlines of drawings, especially such as are to be copied on an enlarged or reduced scale. It consists in first covering the drawing to be copied with a network of lines, then reproducing this network at the scale required, and finally in filling in, by the eye, the portion of the drawing included in each of the reticulations.

335. The Method of Squares applies a similar procedure to the putting into perspective of a complicated perspective plan. A network of lines being first drawn over the plan in question, its perspective representation is easily drawn in parallel perspective. The details of the plan can then be filled in with sufficient accuracy, and the vertical dimensions obtained from a scale of heights.

Fig. 96 illustrates this procedure, *a* being the orthographic plan, *squared*, *b* the perspective plan, and *c* the drawing.

Fig. 96.

The figure does not show how the heights are obtained. They may be obtained either by *squaring* a side elevation and putting it in perspective, after the manner of Fig. 92, *c*, or by erecting lines of vertical measures at convenient points in the plane of the picture, as in Fig. 90.

336. If a sunk perspective plan is used, as in the drawing, the outlines of the plan in the picture can most easily be found by the use of proportional dividers, the distances of the corresponding points from the horizon being proportional.

Mr. Adhémar's Method.

337. The system of Mr. Adhémar is an ingenious variation of the method of Co-ordinates. Like that method, it does not rely upon the use of any vanishing-points except the one at the centre of the picture, and it enables the work, if necessary, to be entirely confined within the limits of the picture itself. Such vanishing-points as lie within these limits, however, whether they belong to any of the lines that occur in the objects represented, or are merely auxiliary, like points of proportional measures or points of half-distance or quarter-distance, are made the most of.

Mr. Adhé-mar's system.

This system, like that described in the previous paragraphs, is especially adapted to cases in which, from the scale of the drawings, from the limitation of the space at command, or from the great distance of the station-point from the picture, the vanishing-points of the principal lines are inaccessible. It leaves one free to take the point of view most conducive to the desired result, without considering whether the making of the drawing will be more or less difficult.

338. In the application of this method Mr. Adhémar employs four special devices. These are Small Scale Data, Vertical Margins, Auxiliary Directrices, and the Inclined Perspective Plan.

339. It is obvious that the larger a perspective drawing is to be made, the more convenient will it be to draw out the data from which it

Small scale data.

is to be constructed at a reduced dimension, a half, a third, or a fourth, of that employed in the picture. Instead, however, of magnifying the data fur- Reduced scales nished by such orthographic drawings in order of height and width in an to bring them up to the scale of the plane of auxiliary plane of meas- measures, as is done with Fig. 84, an auxiliary ures. plane of measures is employed, two, three, or four, times as far off, a plane so distant that the small Reduced scale scale dimensions given in the data can be of depth and fractional points of employed without change to establish scales distance. of height and breadth. The measures of depth, perpendicular to the plane of the picture, are laid off upon the ground-line, as usual; but here, too, the same small scale is employed, the dimensions being transferred to the scale of depth by means of fractional distance-points (142).

340. Fig. 97, *a*, *b*, and *c*, illustrates this procedure. At *a* we have first the elevation of the object Plate XXI. to be drawn, in this case a pyramid; the line Fig. 97. of the horizon is drawn to show how much of the pyramid is above, how much below, the eye. Alongside is an Orthographic Plan, on the same scale, showing the position of the pyramid relatively to the plane of the picture and to a line, G O, drawn upon the horizontal plane perpendicular to the picture, at a convenient distance from the pyramid, to serve as a scale of depth. The dotted lines show the distance from these two lines to the four angles of the pyramid which are visible from the station-point. They are the horizontal co-ordinates of these points. The vertical co-ordinates of these points are

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shown by a dotted line at the side of the elevation. C is, as usual, the centre of the picture opposite the eye, at S, and D$\frac{1}{2}$ is the point of half-distance.

341. At b we have this Orthographic Plan put into perspective, and the pyramid in perspective above it. But the scale is doubled, the distance $g\ c$, upon the ground-line being double that of G C in the plan. Instead of using the ground-line, however, as a line of horizontal measures, or scale of widths, as we have hitherto done, an auxiliary line of measures, $o\ m$, is drawn, at such a distance behind the ground-line as to make $o\ c'$ equal to G C. At o is erected a line of vertical measures, or scale of heights. Upon these lines the dimensions of width and height given in the plan and elevation are set off without change of scale. The dimensions of depth, given on the line G O, are set off at the same scale upon the ground-line, from the point g, and are transferred to the scale of depths, $g\ o$, by means of the point of half-distance, D$\frac{1}{2}$. If the scale were enlarged three times, a point of one-third-distance, D$\frac{1}{3}$, would be employed.

Fig. 97, b.

342. These three scales, or lines of measures, are what are called in geometry co-ordinate axes, and their point of meeting, o, is called the origin of co-ordinates, being the point from which the co-ordinates of any point are measured. These being laid off upon these axes, as explained in the previous paragraph, the position of any point in space is easily ascertained by drawing from the corresponding points on each axis lines parallel to the other axes, in each plane of pro-

jection ; this gives the projection of the point in each
of the three planes. Lines drawn from each of these
points of projection, perpendicular to the plane in which
it lies, and parallel to the other axis, will meet in space
in a point which is the perspective required. As two of
these axes are parallel to the picture, lines parallel to
them are drawn parallel to their perspectives, while the
third axis, being perpendicular to the picture, is, together
with the lines parallel to it, directed to the centre, C.
The perspective of the point 4, the vertex of the pyra-
mid, is ascertained in this way, its distance from the
origin, o, in each direction, being first marked upon the
three axes, or lines of measures ; its projection upon
the three planes is then found ; and finally the point
itself is found where the three co-ordinates in space
meet. The point 4 is seen to be the front upper right-
hand corner of a parallelopiped, of which the origin, o,
is the lower back left-hand corner.

343. But it is obvious that, as the intersection of two
of these lines would suffice to determine the point, the
projection of the point upon two planes is all that is
required. The perspectives of the points 1, 2, and 3 are
accordingly determined only upon the horizontal plane,
and upon the vertical plane on the left, perpendicular to
the picture.

It is worth while to point out that in the perspective
plan the resemblance between the converging co-ordi-
nates, and the edges of the pyramid above, is accidental.
These lines would converge upon C, whatever the shape
of the object to be drawn.

344. Fig. 97, c, shows that by omitting to sink the
perspective plan these operations may all be
Fig. 97, c.   conducted almost within the limits of the
picture itself. It also shows that if we omit the let-
ters and figures, and draw in only so much of the con-
structive lines as are actually necessary to determine
points required, the work is simple and easy, and does
not fill up the space devoted to the picture.

345. We have hitherto regarded the plane of the
picture as a surface of indefinite extent, the exact limits
of the picture itself being the last thing to be consid-
ered. In the system under consideration, however, the
Vertical   shape and size of the picture are determined
margins.   beforehand, its lower limit being set at the
ground-line, from which, on either side of the picture,
arise vertical lines, which are its margins on the right
and left. The points where they stand being noted on
the orthographic plan, horizontal boundary-lines drawn
through these points and the station point enclose a
trapezoidal surface, which is the area seen in the picture.
These horizontal boundaries, radiating from the station-
point, have their perspectives vertical, in accordance
with the nature of converging lines (231). Their per-
spectives are the vertical margins of the picture, which
to the spectator at the station point seem to cover
and coincide with the horizontal boundaries. All
points, accordingly, situated on the horizontal bounda-
ries of the trapezoidal area will appear in perspective
upon the vertical margins of the picture. These ver-

tical margins are not necessarily set at equal distances from the centre, C.

346. If, for instance, the distance of the centre, C, from one of the vertical margins of the picture being determined, the distance of the station-point, S, in front of the picture, is three or four times that distance, then the point of one-third or one-quarter distance, $D\frac{1}{3}$ or $D\frac{1}{4}$, will fall exactly upon that vertical margin. If, also, any lines of the plan are prolonged until they cut the boundaries of the visible horizontal plane, these points of intersection will also appear in perspective upon the vertical margins of the picture. Such lines may accordingly be put into perspective without finding any points except just at the edges of the picture, and upon a line of depths.

347. Fig. 98 illustrates this procedure, showing how the same perspective plan as that determined in Fig. 97 can thus be obtained. At a we

Fig. 98.

have, as before, a small-scale orthographic plan. Assuming $p\,p'$, in the picture plane, as the right and left-hand limits of the picture, diverging lines drawn through these points from the station-point, S, establish the horizontal boundaries of the trapezoidal area. If, now, the sides and diagonals of the plan of the pyramid be extended till they cut these boundaries on either side and the ground-line in front, we shall have the points numbered 1, 2, 3, 4, 5, 6, 7, and 8, their projections upon the line of depths, drawn at right angles to p, being figured 1′, 2′, 3′, 7′, 8′. If, now, we double the scale, as in the previous case, the points 1′, 2′, 3′, 7′, and 8′, are easily found

upon the scale of depths, as before, and 1, 2, 3, 7, and 8, at the same levels upon the vertical margins. The points 4′, 5′, and 6,′ being taken on the small-scale line of widths serve to determine the points 4, 5, and 6 upon the ground-line.

348. The perspectives of these eight points being thus ascertained, it is easy, by drawing the lines 1, 6 ; 2, 7 ; and 4, 8, to draw the perspective plan of the visible half of the pyramid. The point 9, the projection of its vertex, is got by drawing its projection upon the ground-line in the orthographic plan. Indeed, in practice, this method and that of co-ordinates are used interchangeably, as the conditions of each case may render advisable.

349. In Mr. Adhémar's treatise the trapezoidal figure, with its horizontal boundary-lines, is drawn in the ortho- graphic plan, and the corresponding vertical margins are drawn at the edges of the picture, even in cases where the special device described in the previous para- graph is not employed. They serve to define the limits of the work, and it is often convenient, even when one is employing the ordinary method of co-ordinates with the origin of co-ordinates in the plane of the picture, to have the line of depths, *i. e.*, the line of perpendicular or normal measures, and the ground-line, or line of hori- zontal measures, meet in the line of vertical measures, which in that case coincides with the vertical margin in the corner of the picture. This point is then the origin of co-ordinates, as happens in Fig. 92. In em- ploying the method of small-scale data also, although

the origin of co-ordinates is at a distance, it is often convenient to have the ground-line, the line of depths, and the vertical margin, meet at a point.

350. But that this is a mere matter of convenience, which involves no principle, and really has no effect upon the method of working out the perspective problem, may be seen in Figs. 99 and 100, which illustrate the relation between the vertical margins of a picture and the system of small-scale data, according to which they are made.

Vertical margins not necessary with small-scale data.

351. Fig. 99 shows an orthographic plan, on a small scale, with the station-point, S, the point a, of which the perspective is to be found, and the plane of the picture, at G L, at right angles to the Axis, with the centre of the picture at C, and the point of distance at D, C D being equal to S C. The line of depth, G O′ O″, touches the ground-line at G, and the left-hand boundary of the visible area is drawn from S through this point. But as this line plays no part in the solution of the problem its position is immaterial, and it might just as well have been directed more to the left, as shown by the dotted line. The right-hand boundary is also drawn at pleasure. Being drawn, they fix the width of the picture, G L.

Fig. 99, A and B.

Fig. 99, B, shows the same points in the side view, $p\,p$ being the plane of the picture, a the point to be put into perspective, and $a\,g$ the horizontal plane on which it lies, cutting the plane of the picture at the ground-line, g. C g, equal to $c\,a$, shows how far the point in question lies below the level of the eye at S.

352. Fig. 99, C, shows the picture itself on a scale

Fig. 99, C three times that of the plan, C being the centre, and D⅓ the point of one-third distance. The origin of co-ordinates, o, is found by setting off C i, equal to C g in Fig. 99 B, and $i\,o$, equal to C G in Fig. 99 A. The lines $o\,z$, $o\,e$, and $o\,g$, are then the three co-ordinate axes. The ground-line, $g\,l$, is three times as far from the Horizon as in Fig. 99, B.

353. As the scale employed in these operations is

The plane of measures. one third of the scale to be employed in the picture, it follows that the plane of measures containing C i, o and e is three times as far from the eye as is the picture. In the orthographic plan it lies at O'. This distance, G O', of which $o\,g$ is the perspective, may be found directly by drawing a line from D⅓ through o until it intersects the ground-line at o'. The distance, $g\,o'$, gives the distance of o behind the picture, on the scale of the plan. It is equal to G O'. This gives the position of the plane of measures (93), the plane in which these dimensions are taken.

354. The horizontal co-ordinates of any point in the perspective plan can now be ascertained. Let a be such a point. Its horizontal co-ordinate, $u\,a$, equal to $o'\,u'$, being laid off on the horizontal axis from o, gives the point e on that axis, and the perspective of the point a will lie in the normal perspective line drawn through e from the centre, C. The point u upon the line of depths, $o\,g$, may be ascertained either by laying off upon that line the perspective of the distance, O' u, from o, or the distance, G u, from g. Either of these

distances being measured upon the ground-line, from g or from o', determines the point u', which may then be transferred to the line of depth, at u, by means of the point of distance, D$\frac{1}{3}$, A horizontal line from u will cut the normal line through c at the desired point, a.

If the desired point lay above the ground-plane, its height could be set off at the same scale upon a vertical ordinate erected at e, and a normal line drawn through C, and the point thus determined would cut an ordinate erected at a at the required point.

355. It will be noticed that all these operations are quite independent of the limits to be given to the picture, and would have gone on in exactly the same way if the vertical margin had been drawn to the left of the point g, just as the lower margin is independent of the ground-line. Moreover, most of these operations are quite independent of the *scale* to be employed, — that is to say, of the number of times the picture is to be magnified. This is determinéd solely by the distance at which the ground-line is set below the Horizon.

356. This is illustrated in Fig. 99, D, which is identical with Fig. 99, C, except that the line $g\,l$ is between five and six times as far from the Horizon as is the horizontal axis, $o\,e$, instead of three times. This gives upon the ground-line the distance, $g\,o''$, and upon the orthographic plan to point O″, for the position of the plane of measures.

Fig. 99, D.

The fractional point of distance remains unchanged, and the point u' is as far from g as before; u and a are then easily found. If the dimensions of the picture are enlarged proportionally, as is here done, a will be found to occupy the same relative position in the larger picture that it does in the smaller picture above.

It is not of course necessary to know how large a magnifying power one is employing. The ground-line, or the vertical margins, may be set wherever seems best. The rest will take care of itself.

357. Fig. 100 shows that the results attained are also Fig. 100, A and B. quite independent of the size and scale of the picture shown in the orthographic plan. We have at A and B exactly the same conditions as in the previous figure, except that the plane of the picture is further from the spectator, and nearer the point a. The picture there indicated is accordingly larger, the point of distance further from the centre, and the distance, G u, taken on the line of depth, just so much smaller. This line, being still drawn to the edge of the picture, is further from the axis, and the dimension, G C, is greater than before.

In Fig. 100, C, however, we obtain exactly the same Fig. 100, C. result as in Fig. 99, C, the scale of the enlarged drawing being the same.

In this figure the ground-line meets the left-hand vertical margin just in the corner of the picture. The original, at A, being larger than in the previous figure, does not need to be magnified as many times, hardly

more than twice, and the ground-line is accordingly set nearer the Horizon.

In Fig. 100, D, the same result is a third time attained, the line of depth not being brought to the corner of the picture, but set just as far from the axis as in Fig. 99, the point of distance, however, being the same as in Fig. 100, C. Fig. 100, D.

358. It appears from these examples that, if it were for any reason convenient to employ different systems of ground-lines and points of distance, or different lines of depth, in the same picture, one part of it being put in by one system and another by another, there would be no discrepancy in the results. We shall have occasion to avail ourselves of this before we get to the end of the chapter. (375.)

359. We have seen that the method of co-ordinates is specially useful in putting into perspective Auxiliary directrices. irregular figures, figures that have to be treated Mouldings. as a series of points. In drawing mouldings, for instance, especially when the scale of the drawing is so large that they have to be determined with great precision, the system of co-ordinates enables us to fix the horizontal and vertical position of as many points as we please, and thus to attain any degree of exactness in the profiling that we may desire. A single profile being ascertained, any length of moulding may be drawn by drawing lines through the points thus determined to the proper vanishing-points.

360. Fig. 101 shows how the horizontal and vertical

dimensions that define the outline of a cornice may be determined in either of three ways. At a is shown the section of a cornice, with the horizontal and vertical co-ordinates of the principal points. At b this section is put into perspective by means of the perspective plan R above and the point of distance, D^R, and the parallel lines of the cornice are drawn through the points thus determined. At c the perspective of the cornice at the corner, "on the mitre," is ascertained in like manner, with the aid of the vanishing-point of 45°, V^X. This profile serves for the cornice on the right as well as for that on the left. At d the section parallel to the picture is ascertained, and employed for a cornice going off to the right. Being used parallel to the picture, it is put into perspective without change.

Mr. Adhémar employs all these profiles at once, as auxiliary to each other, thus dispensing with both vanishing-points, as may be seen at e. The relative position of these sections may be seen in the perspective plan at the top.

361. This device is of special service in putting circu-

lar mouldings into perspective, as may be seen in Fig. 102, which is borrowed, with the omission of the construction lines, from the plates of his treatise. The construction of these profiles is very easy, and as many of them can be employed as seems necessary.

362. In very large drawings the use of auxiliary profiles as directrices is of special advantage, because, even when the vanishing-points are within reach, the

necessary rulers and straight edges are so long and so flexible that it is difficult to keep the lines true and the mouldings properly proportioned throughout.

363. It has already been said that the reason why the method of the Perspective Plan has attained so little vogue, and has generally been set aside for the method of Direct Projection, is this : that the lines of the perspective plan generally meet at so acute an angle as to make it difficult to determine their exact point of intersection. This objection is overcome by the device of sinking the plan, so that the perspective lines may intersect more nearly at right angles. But however satisfactorily this may be arranged for the foreground of the plan, lines in the more distant parts must always grow gradually more nearly parallel to the horizon and to each other, so that in just that part of the drawing where the scale is the smallest, and precision, accordingly, most important, precision is the most difficult to obtain.

Remote objects.

364. To remedy this evil Mr. Adhémar proposes that the more distant objects, or the more distant portions of an object, shall be projected not upon a horizontal plane but upon an inclined plane, thus increasing the vertical distances and the angles at which the lines of the plan intersect with one another. Fig. 103 shows the device in question. At A is shown the station-point, S, the plane of the picture, $p\,p'$, seen in profile, and the elevation, or edge, of a horizontal ring tangent to the

The Inclined Perspective Plan.
Fig 101.
Plate XXII.

Fig. 103, A.

plane of the picture at p, and the projection of this ring
upon the ground-plan at $g\,g$. This will of course be
a circle, and its perspective, as seen from S, will be an
ellipse, the major axis of which is horizontal and rather
less than the diameter of the circle, and the minor axis
vertical, and equal to the distance, $d\,g$. This ellipse is
shown at a, where is shown also the ground-line, $g\,l$,
the centre, C, the major axis of the ellipse at $f\,f$, and
the horizontal diameter of the circle, which appears as
a horizontal chord of the ellipse at $e\,e$, the further half
of the circle appearing, of course, smaller than the nearer
half. The centre of the circle is seen, in the plane of
the picture, at e.

365. Mr. Adhémar's device for making the perspec-
tive of the further half of the circle as wide as that of
the nearer half is shown at B. The further half of the
ring is projected not upon a horizontal but upon an

Fig. 103, B.
inclined plane. This plane is shown in the
figure, cutting the plane of the picture in a
new ground-line, at g', the horizon of this plane being
not at C but at C'. Using this new horizon and this
new ground-line, the projection of the further half of the
ring can be drawn, as shown at b, with any desired pre-
cision, the same as upon any inclined plane.

But this can be equally well and more simply accom-
plished, merely by sinking to a still lower level the fur-
ther half of the circle in which the ring is projected, that

Fig. 103, C.
is to say, by projecting the further half of the
ring not upon an inclined plane but upon a
lower horizontal plane, as is done in Fig. 103, C. This,

to be sure, though it widens the projection of the further parts as much as may be desired, brings their perspective lower down than that of the nearer parts, as at *c*, which is awkward. But this can be got over, by not only sinking that portion of the perspective plan, but also simultaneously raising the station-point. If this is done judiciously, as at D, the two portions of the perspective plan will preserve their rela- Fig. 103, D. tive position, as at *d*. The next figure shows how this may be effected (370).

366. The results reached in B and D are substantially alike, and it is plain that sinking the The sunk remote portions of the perspective plan, in- plane preferable to the inclined stead of using an inclined plane of projection, plane. accomplishes the practical end in view, by what seems to be a simpler method. It is worth while to point out, however, before leaving the subject, that the results in the two cases are not only similar but identical ; that is to say that the perspective of the projection upon the sunken horizontal plane coincides with that of the projection upon the inclined plane, not only at its extremities but at every point.

367. This may be seen in Fig. 104, where the line A G, arbitrarily divided into two parts at the The results identical. point B is seen projected both upon a hori- Fig. 104. zontal plane at *a*, *b*, and *g*, and upon an inclined plane at *a'*, *b'*, and *g*. The figure shows that if the new station-point S', immediately above S, is so taken that the perspective of *a* as seen from S coincides with that of *a'* as seen from S, the point *g* common to both projec-

tions being its own perspective, then the perspective of b as seen from S′ will coincide with that of $b′$ as seen from S. That is to say, if the perspectives of the two projections coincide at their extreme points they will also coincide at every intermediate point.

368. The proof of this follows from the relations of the lines of the figure, in which perpendiculars let fall from the two station-points S and S′ give the corresponding Horizons at C and C′. For, in the first place, since the trapezoids S S′ C′ $a″$ and g a $a′$ $a″$ are divided by the line a S′ into similar triangles, the trapezoids themselves must be similar, and since their homologous sides are parallel their diagonals S C′ and g $a′$ are also parallel. The right-angled triangles S S′ C′ and b $b′$ g, having their vertices upon the line C′ g are accordingly similar, and their heights, S′ C′ and b g, are proportional to their bases S S′ and b $b′$. But the scalene triangles S S′ $b″$ and b $b′$ $b″$ are also similar, and their heights also are proportional to their bases S S′ and b $b′$, which are the same as those of the right-angled triangles. Their heights must accordingly be the same as the heights of the right-angled triangles, and their vertices at $b″$ must also fall upon the line C′ g. That is to say, the perspective of $b′$ as seen from S coincides with that of b as seen from S′, and this is true wherever the transverse line b $b′$ may be drawn.

369. Since, as has been shown, S C′ and g $a′$ are parallel whenever S $a′$ and S′ a intersect in the plane of the picture, it follows that the point C′, which marks the position of the Horizon of the horizontal plane of

projection $a\,b\,g$, when the eye is at S′, marks also the horizon of the inclined plane of projection $a'\,b'\,g$ when the eye is at S, and that g marks the ground-line of both the sunk plane and the inclined plane.

370. This enables us to answer a question raised in regard to Fig. 103, D and d (367), and to say just how far the Horizon must be raised and the ground-line lowered in order to make the perspectives of successive sunk planes continuous, one with another, like the perspectives of successive inclined planes. The successive horizons and ground-lines of the horizontal planes are to be drawn just where the successive horizons and ground-lines of the inclined planes would be, the distance that the station-point is raised being proportional to its distance in front of the picture, and the distance that the horizontal plane is lowered being proportional to its distance behind the picture.

371. The identity of the results obtained by these two processes is again illustrated in Fig. 105, a, b, c, and d. At a is the orthographic projection of a circle, tangent to the plane of the picture, G L. The area of this circle is divided into four segments by planes which cut the line of depth at the points A, B, D, E, and G. In the Fig. 105, c, above, these segments are shown projected upon four horizontal planes having their ground-lines at $g, g''\, g'''$, and g^{IV}, the corresponding station-points, S, S″, S‴, and S^{IV}, being so adjusted, in

16

accordance with the relations shown in Fig. 104, that
the perspectives of the plans of the different segments
form a continuous figure, the perspective of the further
edge of each segment, as seen from the lower station-
point, coinciding with that of the hither edge of the
segment below, as seen from the station-point next
above.

372. Alongside, to the right, drawn to the same scale,
is the perspective plan of the circle. The left side shows
the perspective of its projection upon the ground-plane,
$a\,g$. This is an ellipse, divided into the four segments cor-
responding to those into which the plan is divided. The
right-hand side shows the perspectives of the projections
upon the three sunken planes, which have their ground-
lines at g'', g''', and g^{IV}, as they appear when viewed,
respectively, from S'', S''', and S^{IV}. The second seg-
ment of the second ellipse, the third of the third, and
the fourth of the fourth, are extended so as to be as
wide as the first segment of the first, and, like it, are
drawn in with a full line, thus making a continuous
figure, in which the remoter parts of the circular surface
are as advantageously displayed as the nearer parts.

373. In Fig. 105, b, we see that exactly the same
Fig. 105, b.   result is reached if, instead of thus using suc-
Successive   cessive horizontal planes, we employ Mr. Ad-
inclined
planes.   hémar's device of successive inclined planes.
Not only in this case do the perspectives themselves
coincide at all points, but the ground-lines and horizons
of the inclined planes are identical with those of the

horizontal planes in the previous figure.   The practical
work of thus constructing the perspective drawing is
then the same, whichever method we prefer to employ.
In either, the dimensions of width and height may be
laid off, according to the scale of the picture, upon the
successive ground-lines and upon the vertical margins.
The dimension of depth in each plane may also be laid
off upon the same ground-line, and transferred to the
line of depths by the successive points of distance,
which, upon each Horizon, will give the common dis-
tance of the successive station-points from the picture.

374. This is illustrated in the right-hand side of
Fig. 105, d, which shows the successive per-
spective plans figured in Fig. 105, b, upon a   Fig. 105, d.
scale double that of the previous figures.   In two of
the four segments shown in plan in Fig. 105, a, a point
is taken at random, and its co-ordinates, measured from
the ground-line and from a line of depths erected at L,
are shown.   These points, numbered 1 and 4, in the first
and fourth planes, are put into perspective by means of
the successive ground-lines $g\,l$ and $g^{IV}\,l^{IV}$, and the points
of half-distance, $D\frac{1}{2}$ and $D\frac{1}{2}^{IV}$.   Any number of points
of the circle, or of any other figure, could be put into
perspective upon the successive planes in the same way,
just as in Fig. 92, Plate XX. (330).

375. These figures show also that if, in either figure,
105, b, or c, we consider the vertical planes   The results
which divide the plan into segments to be so   identical.
many successive planes of measures (93) intersecting
the four successive ground-planes, whether horizontal

or inclined, in the ground-lines g_2l_2, g_3l_3, g_4l_4, the per-
spectives of the successive segments will form a consecutive series. Whether viewed from S in Fig. 105, b,
or from the successive station-points of 105, c, they will
present the appearance of Fig. 105, d. This is an additional illustration of the point made in a preceding
paragraph (358), and illustrated in Figs. 99 and 100 of
the previous plate, that when a perspective drawing is
enlarged to a given scale it makes no difference in the
result at what distance from the station-point the plane
of measures in the orthographic plan is set, and that, if
convenient, part of a picture may be put in by means of
one such plane, with its corresponding ground-line, line
of depths, and point of distance, and part by means of
another.

376. This also is illustrated in Fig. 105, d, in which
Fig. 105, d. the methods of Figs. 99 and 100 are applied
Small-scale
data. to produce exactly the same outline upon the
left-hand side of the figure by the method of small-
scale data as has been produced on the right-hand side
by the method of co-ordinates (374). The figure shows
the successive ground-lines, $g\,l$, g_2l_2, g_3l_3, g_4l_4, and the
successive lines of depth, $g\,C$, g_2C'', g_3C''', and g_4C^{IV}.
The successive origins of co-ordinates, o, o_2, o_3, and o_4,
are determined as in Figs. 99 and 100, and the small-
scale data taken from the orthographic plan are laid
off from them by aid of the successive fractional points
of distance, D_1, D_2, D_3, and D_4. The co-ordinates of
the points 1, 2, 3, and 4, on the left-hand side of the
circle, corresponding to those already found on the right-

hand side, are thus used in the figure, and their perspectives found.

377. This process, also, is accordingly equally available, and is identical in its methods and results, whether we suppose the successive ground-planes to be inclined, as Mr. Adhémar has them, or to be horizontal, as we have suggested, and as seems rather more in harmony with the conceptions of this treatise, and with the habitual use of the sunk perspective plan to effect the same ends.

378. We have seen, in Fig. 103, *c* (365), that if we constantly lower the ground-plane so as to get a better view of its more remote parts, we must at the same time raise the station-point and the Horizon. Otherwise, the perspective plan will be dislocated, as shown in Fig. 101, *d*.

In practice the Horizon need not be raised.

But, in practice, this stretching out of the more remote portion of a plan is generally needed not in its whole extent, but only in part, often only for particular objects. Such isolated details are much more simply and conveniently treated by the use of the ordinary sunk horizontal plan, without moving the station-point. The advantage of this method in getting such auxiliary operations out of the way of the picture has frequently been illustrated in these pages; as, for example, in Fig. 20, Plate VII.

379. Figs. 106, 107, 108, and 109, Plate XXIII., show the application of this method to the interior of the *Halle aux Blés*, in Paris. The main lines of these figures are taken from Mr. Adhémar's treatise, though the construction lines are some-

Plate XXIII. Figs. 106, 107, 108, and 109.

what differently adjusted, and the notation altered. It is hardly necessary, after what has been said, to enter into a detailed explanation of these examples. They reproduce the main features of Figs. 99 and 100, as well as of Figs. 103 and 105. Two points, a and b, in the second and third planes respectively, are ascertained by the use of small-scale data.

CHAPTER XVI.

THE INVERSE PROCESS.

380. It is sometimes desirable to invert the proced-
ure described in the preceding chapters. Instead of
beginning with an orthographic plan and elevation, de-
riving thence a perspective plan and perhaps, Given the
perspective
as in the previous chapter, a perspective eleva- to find the
real form and
tion, and then finally arriving at a complete dimensions.
perspective drawing, it is often possible by a reverse
process to derive the perspective plan and elevation from
the drawing, and from them to obtain the actual shape of
the object, and its relations to the spectator and the plane
of the picture. Its dimensions can also be determined
if the dimensions of certain lines in it are known.

To effect this with any approach to precision it is
necessary that the perspective lines shall make a suffi-
ciently large angle with each other or with the Horizon
clearly to indicate the position of the vanishing-points;
that is to say, the object shown must either be large,
near at hand, or considerably above or below the eye.

381. If the object is in oblique, or three-point per-
spective, and its three vanishing-points can be Oblique, or
three-point
fixed with precision, there is no difficulty, as perspective.
has been shown in Chapter VII., in determining the posi-
tion of the spectator. This fixes the centre of the picture,

the station-point, the distance of the station-point from the centre and from each of the vanishing-points, and all the points of distance. For lines connecting the three vanishing-points represent the three horizons, and the meeting-point of the perpendiculars let fall from the angles of the triangle thus formed upon the opposite sides, is the centre, C; the distance of the station-point in front of this point, and its distance from each of the vanishing-points, is then easily determined (168), and the points of distance found. If, then, the length of any of the vanishing lines is known, a line of measures parallel to one of the horizons can be drawn through one of its extremities, its true dimension according to the scale of the drawing, or of that part of it, found by means of a point of distance, and the scale of the draw-ing and the dimensions of every other part ascertained.

382. This is illustrated in Plate XXIV., Fig. 110. If Plate XXIV. we suppose the perspective of the rectangular Fig. 110. block to be given, the vanishing-points V^L, V^M, and V^O, and the traces that connect them, can be obtained, the ground-lines, or lines of measures $o\,m$, $l\,m$, and, if necessary, $l\,o$, drawn, and the relative dimensions of the edges determined. Whether the block is large or small cannot of course be learned. The size of the miniature block, supposed to be in contact with the picture at a, is determined, its edges being equal to $a\,l$, $a\,m$, and $a\,o$; but there is no means of knowing how much larger the block itself is than this miniature rep-resentative. To determine this we must know either the actual dimensions of one edge of the block, or the

distance of the block behind the picture. Neither of these can be shown by the picture itself.

383. If the object is drawn in two-point, or angular perspective, as is generally the case, it does not suffice for the determination of its shape, position, and relations to the spectator that the vanishing-points of its principal lines should be known. For fixing two vanishing-points does not, as fixing three does, determine the position of the spectator and of the centre of the picture, and thence of the points of distance, nor does it determine the *attitude* of the object, or the angles its sides make with the plane of the picture. Fixing the vanishing-points only restricts the *locus* of the spectator's position to the semicircle subtended by the line joining them; they determine neither the attitude of the object nor its shape. In Fig. 111, for instance, we have at A and B the same perspective and the same vanishing-points. But at A the station-point, S, and the centre, C, are assumed to be well over towards the right, and at B towards the left. The perspective plans and the elevations derived from them are shown below. The plans are alike, but the points of distance being different the dimensions found upon the ground-lines are different, and the proportions of the building and the slope of the roof come out differently. But while the buildings, though differing in size and shape, are alike in perspective, the doors and windows, which are of the same size and shape in one building as in the other, come out differently in perspective.

384. In order to interpret correctly a drawing made
in angular, or two-point perspective, it is
necessary to have definite information as to the
position either of the centre, C, of the vanishing-point
at 45°, V^X, or of one of the points of distance, D^R
or D^L. The centre is generally nearly in the middle of
the picture, but that it is exactly there is not to be
taken for granted. Its position is often precisely indi-
cated, however, by some secondary object, which is
drawn in parallel perspective; and it is always a good
plan to introduce some such object as the pile of boards
in Fig. 111, A, as a guide to the spectator.

385. It often happens, however, especially in archi-
tectural drawings, that the nature of the sub-
ject is such as to furnish diagonal lines lying
at 45° with the principal directions, lines that
we have called X dividing the angle made by the lines
R and L. Fig. 112 shows, by means of a little ele-
mentary geometry, how in this case the station-point, S,
is to be found, V^R, V^L, and V^X being given. As the
angle $V^R S V^X$ is an inscribed angle of 45°, its sides
must include an arc of 90°. A line drawn from x in
the figure through V^X to the opposite circumference
fixes the position of S, and hence of C, D^R, and D^L.

386. It is not often that the position of either of the
points of distance can be detected by mere
inspection of the picture. But it often hap-
pens that the real or proportional dimensions
of some of the lines in the picture are known.
In that case one of the points of distance can be ascer-

tained, and the other elements of the problem then easily determined.

Let us suppose, for instance, in Fig. 113, that the rise and tread of the steps are known to be six and twelve inches. A line of equal measures, $l\,r$, laid off parallel to the Horizon, from the front edge of the first step, in length equal to twice the vertical edge, forms, with the horizontal line in perspective, and a third line, joining their further ends, the three sides of an isosceles tri- angle. The vanishing-point of the third line, the base of the triangle, gives the point of distance, D^R. The distance from this point to its corresponding vanishing-point is the dis- tance of the station-point from that vanishing-point; that is to say, $D^R\,V^R = V^R S$. S, which must lie somewhere in the semicircle of which $V^L\,V^R$ is the diameter, is then easily found, as in the figure. C, D^L, and V^X, immediately follow.

The point of distance being given to find S, C, etc.

Fig. 113.

387. When the drawing to be interpreted is made in parallel perspective it is generally easy enough to find the centre of the picture, the vanish- ing-point of the lines perpendicular to it. But, as in the previous case, it is impossible to tell what is the real shape of the objects represented, or to know from what distance the picture should be looked at, unless the real shape of some one of the objects is known inde- pendently. If Fig. 20, for instance, Plate VI., is looked at from a point about three inches in front of C, as may be done by looking at it through a pin-hole, so as to

One-point, or parallel perspective.

obtain a clear image on the retina, the little pavilion represented looks about square, the steps on the side seeming very steep. Seen from a distance of several feet it looks two or three times as long as it is wide, and the steps seem of very easy grade. The posts at the corners are presumably square, and the lines of the pavement and of the hips of the roof, in plan, are presumably directed to the vanishing-point of 45°, which is the point of distance; and the steps have presumably the same slope as those on the right. The point of distance can be found by the same method as in the preceding paragraph, and the true shape of all the objects in the picture determined.

388. Of course these results are based upon the understanding that the objects represented are rectan-

Acute and obtuse angles. gular. If the lines that define them are known

Fig. 114. to form acute or obtuse angles, instead of angles of 90°, the line joining their vanishing-points must be treated as the chord of a circle instead of as a diameter, as is done in Fig. 114.

At A is shown an obelisk in perspective, presumably

Fig. 114, A. square in plan. The methods described in the previous paragraphs suffice to determine successively the principal vanishing-points, V^R, V^L, and V^X, the centre, C, the points of distance, D^R and D^L, and a perspective plan. The dimensions can then be determined, according to the scale of the drawing, and that scale may be determined if any one of the dimensions is known.

At Fig. 114, B, is another drawing, the horizontal and vertical lines of which are identical with the first. But this obelisk is understood to be triangular and equilateral, with angles of 60° instead of 90°. A perspective plan, with the vanishing-points V^R and V^L, are easily determined, as is also V^X, the vanishing-point of the line bisecting the solid angle in contact with the picture. These elements suffice to determine the orthographic plan, Fig. 114, C. As the angle at the station-point, S, is only 60°, in place of 90°, it is included in an arc of 240°, the point S lying somewhere in that arc, which is its *locus.* The point V^X, however, enables us, as the point $V^{X\prime}$ did in the previous case, to fix the exact position of S, by drawing a line through the summit of the arc at X, and the point V^X. If then the eye is placed in front of Fig. 114, A, opposite C', at the distance indicated by S', on the plan below, the obelisk will look square; if it is placed opposite C, in Fig. 114, B, at the distance indicated by S, it will appear to have the section of an equilateral triangle.

389. The little pyramid on top is, however, differently drawn in the two cases, and the position of the apex suffices to show that the upper figure has four sides, the lower but three. The angles at which these sides meet, however, is necessarily intermediate.

390. The fact that acute or obtuse angles can thus be interpreted as right angles makes it difficult to represent them satisfactorily when there is nothing else in the picture to guide the judgment. It often happens in

the case of buildings situated where two streets meet at
an odd angle that drawings of them look as if the build-
ings were square.   To obviate this it is necessary, as
has been said, to introduce something which is unmis-
takably rectangular, such as an awning, or chimney, or
a cart backing up to the sidewalk, like the pile of boards
in Fig. 111, A.

391. If it were not, indeed, for the facility with which
the mind thus gives the most reasonable interpretation to
the phenomena that meet the eye, even the right angles
shown in perspective would generally look either acute
or obtuse, since it is only when the eye is exactly at the
station-point, or rather when it is on the circumference
of a semicircle lying between the vanishing-points, that
the angle really looks as a right angle would.   But these
distortions, like the other distortions that arise from
abandoning the station-point, are made light of by the
intelligence.   It is only in the case of circles, cylinders,
and spheres that one is disturbed by them.   In those
cases, indeed, remaining at the station-point hardly
suffices to reconcile one to the drawing, as has been
explained in Chapter XII.

CHAPTER XVII.

SUMMARY. — PRINCIPLES.

392. LET us now review the ground gone over in the previous chapters, giving the subject a somewhat more formal treatment. We can then proceed to extend the principles and methods employed to the solution of some problems which we have not as yet encountered.

Propositions and Definitions.

393. A perspective drawing is a representation of the lines that would result from the projection upon a transparent plane of a plane or solid figure situated behind that plane, the lines of projection being visual rays converging from every point of the figure to the eye of a spectator situated in front of the plane.

This sort of projection is called Conical Projection, as distinguished from Orthographic Projection, in which the lines of projection are parallel, and are normal to the plane of projection. See Fig. 115, Plate XXV.

Plate XXV., Fig. 115.

The transparent plane of projection is called the Perspective Plane.

The perspective plane.

The position of the spectator is called the Station-

point.

The point in the perspective-plane opposite the station-point is called the Centre. It is the ortho-graphic projection of the station-point upon the per-spective-plane, and is the point in that plane nearest the spectator's eye. It is not necessarily in the middle of the picture.

The line of projection drawn from the station-point normal to the perspective-plane and passing through the Centre, is called the Axis. Its length gives the dis-tance of the eye in front of the perspective-plane.

Unless otherwise expressly mentioned, the perspective-plane is understood to be vertical, the Axis, accordingly, horizontal, and the Centre on a level with the eye at the station-point. The eye is supposed always to remain at the station-point.

394. The conical projection of a figure, line, or point

upon the perspective-plane is called its per-spective. When viewed from the station-point the perspective exactly coincides with and covers the object itself.

An object and its perspective are, accordingly, not to be distinguished in the picture; but it is necessary to distinguish between them in idea, and sometimes, to pre-vent confusion, in language.

Objects lying at the same distance as the perspective-

plane are represented as of their real size. All lines and figures lying in the perspective plane are their own perspectives.

Objects behind the perspective-plane have their perspectives smaller than themselves. Objects in front of the perspective-plane may be conically projected back upon it, and have their perspectives larger than themselves. All objects have their remoter parts drawn to a constantly smaller scale than their nearer parts.

395. The surface upon which the drawing is made is called the plane of the paper, or plane of the picture. See Fig. 116, A.

The plane of the picture. Fig. 116.

When the objects to be drawn are very small, or the picture very large, the plane of the picture and the perspective-plane may coincide, as in Fig. 115.

But generally the objects to be drawn are many feet in dimension, as at B, Fig. 116, while the drawing is to be measured by inches ; and as it is always convenient, for practical reasons, to have the plane of projection as near the object as possible, and consequently of about the same size, while the picture must be small and near at hand, the plane of the picture and the perspective-plane are generally nearly as far from each other as the spectator is from the object, as in Fig. 116.

In this case the drawing is a miniature, or small-scale representation, of the perspective lines supposed to be traced upon the perspective-plane. It may be regarded as a perspective of the perspective-plane, as if made upon a second transparent plane, near the eye, parallel to the perspective-plane. All the lines in the perspective-plane will be represented unchanged in direction, and reduced in a uniform ratio. The scale of

Scale.

the drawing will depend upon the relative distance of the two planes from the eye (94).

396. The same result may be reached by conceiving the drawing to be made not from the object itself, many feet distant and many feet in dimension, but from a miniature, or model, close to the plane of the picture, or even in contact with it. Any line that lies in the plane of the picture is then its own perspective (394). See Fig. 117.

In this case the perspective-plane and the plane of the picture coincide, as in the case of small objects. As this is the most convenient working hypothesis it is generally adopted, and the distinction between the two planes is not recognized.

In the previous chapters the perspective-plane, where it has been necessary to speak of it, has been called the plane of measures (93). In this chapter and the following one the perspective-plane and the plane of the picture will, as usual, be considered identical.

Planes.

397. Every plane figure, whether horizontal, vertical, or inclined, is regarded as lying in and forming a part of an indefinitely extended or infinite plane, which in its hither, or nearer, part intersects the perspective-plane in a line called its initial-line. In its further part it tends on all sides to reach, but can never pass, an infinitely distant line, which is a great circle of the infinite sphere of which the station-point is the centre, and which

Finite and infinite planes. Fig. 118.

is called the vanishing-line, or horizon, of the plane.
See Fig. 118.

The perspective of such an infinite plane extends,
in the perspective plane, from the initial-line, The initial-
which is its own perspective, since it lies in line, hori-
zon, and
the perspective-plane (394), to the perspective trace.
of its horizon, but not beyond it. The perspective of
the horizon of a plane is called in this treatise the *trace*
of a plane.

398. Planes that are parallel to one another are said
to belong to the same system of planes. They Parallel
planes.
have the same horizon, which is called the Fig. 119.
horizon of the system, and its perspective is called the
trace of the system. The initial-lines of parallel planes,
which are the lines in which they cut the plane of the
picture, are naturally parallel to one another. A plane
drawn through the station-point, in front of the picture,
and through the horizon of the system, is a member of
the system seen edgewise. It intersects the perspective-
plane in the trace of the system. The initial-line of
this plane and its trace, or the perspective of its hori-
zon, coincide. See Fig. 119.

The initial-line of a plane is accordingly parallel to
its trace.

399. Hence the line in which a plane passing through
the station-point, and parallel to a given plane, intersects
the perspective-plane, is the trace of the given plane
and of the system of planes to which it belongs. Such
a plane, since it passes through the eye, is seen as a line,
covering and coinciding with its horizon and the per-

spective of its horizon. The horizon of a plane and the trace, or the perspective of this horizon, may accordingly be found by *looking* in directions parallel to this plane.

400. Hence, whatever angle two planes, or two sys-
The angle of tems of planes, make with each other in space,
two planes.
Fig. 120. planes drawn through the station-point to their horizons, or to the traces which are the perspective of their horizons, will make the same angle at the station-point, in the air, in front of the picture, as do the planes themselves. See Fig. 120.

401. A right line lying in a plane is called an element of that plane.

A right line normal or perpendicular to a plane is called an axis of that plane or system of planes.

All planes have an element, or system of elements, parallel to the picture.

All planes have an element, or system of elements, that is horizontal.

402. Planes parallel to the picture have all their ele-
Planes par- ments parallel to the picture. See Fig. 121.
allel to the
picture. The axes of such planes are normal to the
Fig. 121. picture and parallel to the Axis. The initial-lines of such planes, and their traces, are at an infinite distance, and cannot accordingly be represented. Lines and figures in any such plane are drawn at a uniform scale.

All the planes of a system not parallel to the picture,
Planes not whether horizontal, vertical, or oblique, have
parallel to
the picture. the same horizon and the same trace, and
Fig. 119.

their initial-lines are parallel to each other and to the trace of the system, as above. See Fig. 119.

Planes that have elements that are perpendicular or normal to the picture are called normal planes. Normal planes. Their axes are parallel to the picture. The Fig 122. traces of such planes pass through the centre of the picture. Their traces and initial-lines are perpendicular to their axes, and give the true slope and direction of the steepest lines of such planes; that is to say, of the planes themselves. These steepest lines are parallel to the picture. See Fig. 122.

The horizontal elements of normal planes are normal to the picture.

403. Horizontal planes are normal planes in which all the elements are horizontal. Their axes are vertical. The trace and the horizon of horizontal planes Horizontal planes. are both called *the* Horizon. The most im- Fig. 115. portant horizontal plane is called the ground-plane, and its initial-line the ground-line. See Fig. 115.

Other planes besides horizontal planes have only one set of elements horizontal. If this is parallel Inclined planes. to the picture, their traces and initial-lines Fig. 123. will be horizontal; *i. e.*, parallel to the Horizon. These are sometimes called inclined planes. Their axes are inclined also. See Fig. 123.

Other planes, besides those parallel to the picture, have only one set of elements parallel to the Vertical picture. If this is vertical they are called planes. vertical planes. Their traces and initial-lines are ver-

tical, and their axes are horizontal, as shown at A, in Fig. 122.

404. In general, those elements of planes that are
Oblique planes. parallel to the picture are neither vertical nor
Fig. 124. horizontal, and their horizontal elements are
neither parallel nor normal to the picture. Such planes are called oblique planes. Their traces and initial-lines make an angle with the Horizon and the ground-line, but this angle is less than their true slope. Their axes also are oblique, being inclined both to the picture and to the ground-plane. See Fig. 124.

Lines.

405. Every line is regarded as lying in and forming
Finite and infinite lines. part of an indefinitely extended or infinite
Fig. 125. line, which, if not parallel to the perspective-
plane, or plane of the picture, pierces it at its hither, or nearer end, at a point called its initial-point, and at its further end tends to reach, but can never pass, an infinitely distant point, called its vanishing-point. See Fig. 125.

The perspective of such an indefinite line is drawn
The initial-point, vanish-ing-point, and its perspec-tive. in the plane of the picture from the initial-
point, which is its own perspective, towards,
or even up to, the perspective of its vanish-
ing-point, but it can never pass beyond this point. It is the line in which the plane of rays, drawn from the given line to the station-point, intersects the perspective-plane.

406. Lines that are parallel to one another, whether
Parallel lines. horizontal, vertical, or inclined, are said to
Fig. 126. belong to the same system of lines. They

have the same vanishing-point, which is called the vanishing-point of the system. A line from the station-point to this vanishing-point is a member of the system, seen endwise. It pierces the perspective-plane in the perspective of the vanishing-point. Its initial-point and the perspective of its vanishing-point coincide. See Fig. 126.

407. Hence, the point at which a line drawn through the station-point, parallel to a given line, pierces the perspective-plane, is the perspective of the vanishing-point of the given line, and of the system to which the line belongs. Such a line, since it passes through the eye, appears as a point, covering and coinciding with the vanishing-point and its perspective. The vanishing-point of a given line, and the perspective of this vanishing-point, may accordingly be found by *looking* in a direction parallel to that of the line.

408. Hence, whatever angle two lines make with one another in space, lines drawn from the station-point, in front of the picture, to the perspective of their vanishing-points, will make the same angle in the air at the station-point. See Fig. 127.

The angle made by two lines.

Fig. 127.

409. Lines belonging to a system that is parallel to the perspective-plane, whether vertical, horizontal, or inclined, have their perspectives parallel to themselves and to each other, their initial-points and the perspectives of their vanishing-point being alike at an infinite distance upon the perspective-plane. They are drawn to the same scale in all their parts. See Fig. 121.

Lines parallel to the picture.

Such are the axes of normal planes. If such planes are horizontal their axes are vertical, and have their vanishing-points in the zenith and nadir; if vertical, their axes are horizontal; if inclined, their axes are inclined, in a contrary sense. See Fig. 122.

410. The perspectives of a system of lines not parallel to the perspective-plane, whether horizontal or inclined, are not parallel to the lines themselves nor to each other, but converge towards the perspective of their vanishing-point. They are drawn to a constantly diminishing scale in their remoter parts. See Fig. 126.

Lines not parallel to the picture.

Such are the axes of oblique planes, which are oblique also, being inclined both to the picture and to the Horizon. Such, also, are the axes of inclined planes which have their horizontal element parallel to the picture. Their axes also are inclined, in the contrary sense, and have their vanishing-points in a vertical line passing through the Centre. The axes of all vertical planes have their vanishing-points in the Horizon.

411. Lines normal to the perspective-plane, and parallel to the axis, have the centre of the picture for their vanishing-point. See Fig. 128.

Normal lines.
Fig. 128.

Such are the axes of planes parallel to the picture. See Fig. 121.

Lines that lie in the perspective-plane are their own perspectives (394), and are consequently drawn of their own size and shape. See Fig. 117.

Lines in the plane of the picture.

412. All lines lying in or parallel to a plane have their vanishing-points in the horizon, and the Lines in planes. perspectives of their vanishing-points in the Fig. 129. trace of the plane, and the lines that lie in it have their initial-points in its initial-line. See Fig. 129.

[All the elements of the ground-plane have their vanishing-points in the Horizon and their initial-points in the ground-line. See Fig. 115.]

Conversely — The trace, or horizon of a plane, passes through the vanishing-points of all the lines that lie in or are parallel to it, and the initial-line through the initial-points of the lines that lie in it.

[Since two lines are sufficient to determine a plane, this proposition takes, in practice, the following form : —

The trace of a plane passes through the vanishing-points of any *two* lines that lie in it, and its initial line through their initial-points.]

Hence : — The line of intersection of two planes has its initial-point at the intersection of their A line in two planes. initial-lines, and its vanishing-point at the Fig. 130. intersection of their traces, or horizons. See Fig. 130.

Conversely : — The traces or horizons of all the planes that can be passed through or parallel to a given line will intersect each other at its vanishing-point, and the initial-lines of all planes passed through it will intersect each other at its initial-point.

413. Since a plane may be conceived to be passed through a line in any direction, and its trace and its

initial-line may accordingly pass through its vanishing-point and its initial-point in any direction, it follows that any two parallel lines in the perspective-plane, Accidental drawn at random through the initial-point and traces and initial-lines. the vanishing-point of a given line, may be taken to represent the initial-line and the vanishing-line, or trace, of a plane passing through the given line.

Hence : — Any line in a drawing that happens to pass through the initial-point of a perspective-line may be taken as the initial-line of a plane containing the perspective-line, a plane of which the trace, or the perspective of the horizon, may be found by drawing a parallel line through the vanishing-point of the perspective-line ; *conversely*, any line that passes through the vanishing-point may be taken as the trace of a plane, and the corresponding initial-line may be drawn parallel to it through the corresponding initial-point.

414. The perspective of the horizon, or trace, and Lines parallel the initial-line of a plane are, as has been to the trace of a plane. shown, always parallel to each other and to the initial-lines of all the other planes of the system (398).

They are parallel also to the lines in which the plane in question, and all the other planes of the system, intersect all planes parallel to the perspective-plane and to the perspectives of these lines. For the lines in which the two systems of planes intersect are all parallel, and parallel to the picture, and their perspectives

are accordingly parallel to themselves and to each other. They are parallel also to all other lines lying in any of the planes of the systems, which are parallel to the perspective-plane and to their perspectives.

Points.

415. A given point is regarded as lying in a given line at a known distance from its initial- Points in lines. point.   See Fig. 131.   Fig. 131.

In this case the position of the point may be considered to be given by polar co-ordinates, the initial-point being the origin of co-ordinates, and the direction being given by the vanishing-point.

When the given line is normal to the perspective-plane the initial-point is the orthographic projection of the point upon that plane.   In this case the Co-ordinates of a point. position of the point may be considered to be Fig. 132. given by three rectangular co-ordinates, of which the co-ordinates of the initial-point, which lie at right angles to one another in the perspective-plane, are two, and the given line, lying at right angles to them both, the third.   See Fig. 132.

416. The perspective of a point is the point at which a ray drawn from the given point to the eye pierces the perspective-plane.

A point lying at the intersection of two lines has its perspective at the intersection of their A point in two lines, in perspectives.   See Fig. 129, *a*.   the perspec- tive-plane.

A point lying in the perspective-plane is its own perspective.   See Fig. 129, *b*.

The Points of Distance.

417. Through a line, the perspective of which is given
by its initial-point and the perspective of its vanishing-
point, let an auxiliary plane be passed, its initial-line and
The point of trace being drawn parallel to each other, in
distance.
Fig. 133.   any convenient direction (413). See Fig. 133.
They will pass through the initial-point and the perspec-
tive of the vanishing-point of the given line.   Let now
equal distances be assumed upon this initial-line, as at *m*,
and upon the given line itself, — not its perspective, — as
at *a*, measuring from the initial-point. The lines thus set
off will form the two sides of an isosceles triangle I *m a*,
lying in the auxiliary plane, behind the plane of the
picture, having one of its sides, I *m*, in the plane of the
picture.   If now this triangle be completed its base will
be a line also lying in the auxiliary plane, and the
vanishing-point of this base will lie in the horizon of
this plane.   The perspective of this vanishing-point will
be found in the trace of the plane, as at D, in the
figure.   This vanishing-point, which is also the vanish-
ing-point of all lines drawn parallel to the base of the
isosceles triangle, and consequently intercepting equal
portions upon the given line and the initial-line, is called
a point of distance of the given line.

418. The perspective of a point of distance of any line
is a point hardly less important than is the perspective
Its perspec-   of its vanishing-point. The latter serves to
tive.   determine the direction in which to draw the
perspective of a given line ; the former serves to deter.

mine its length. For if, in the plane of the picture, any given length, as I m, be laid off upon the initial-line of the auxiliary plane, and a line be drawn *across* the perspective of the given line towards the perspective of the point of distance, this line will be the perspective of the base of the isosceles triangle, of which the distance, I m, taken upon the initial- The isosceles triangle behind the picture.

line is one side, and the distance, intercepted upon the given perspective-line is the perspective of the other side. An indefinite line, then, being given by its perspective, any required length can be determined upon it by setting off this length upon such an initial-line and transferring it to the perspective of the given line by means of the perspective of a point of distance.

419. The perspective of the point of distance will lie in the trace of the auxiliary plane, as at D, as will also the perspective of the vanishing-point, as at V. These two points, which lie in the plane of the picture, and the station-point, S, which is in the air in front of the picture, form the vertices of a triangle lying in front of the plane of the The isosceles triangle in front of the picture.

picture, of which one side, V D, lies in that plane and the other two meet at the spectator's eye, or the station-point. One side of this triangle lies in the trace of the auxiliary plane, and is accordingly parallel to that side of the isosceles triangle, behind the picture, which lies in the initial-line of the auxiliary plane; *i. e.*, V D is parallel to I m; the side that extends from the station-point to the perspective of the vanishing-point of the

given line is parallel to that side of the isosceles tri-
angle which lies in the given line; *i. e.*, S V is parallel
to I *a ;* and the side that extends from the station-point
to the perspective of the point of distance is parallel
to the base of the isosceles triangle; *i. e.*, S D is par-
allel to *m a*.   The triangle in front of the plane of the
picture will accordingly also be isosceles, the homolo-
gous sides being parallel to the sides of the isosceles
triangle behind the picture, and *the perspective of the
point of distance will be as far from the perspective of the
vanishing-point as the perspective of the vanishing-point
is from the station-point ; i. e.*, V D is equal to S V.

420. This gives the following easy rule for finding the
perspective of a point of distance : —

Given the perspective of a line by its initial-point

To find a
point of
distance.   and the perspective of its vanishing-point, —
find the distance from the station-point in
front of the picture to the perspective of the vanish-
ing-point, and lay off this distance from the perspec-
tive of the vanishing-point along any line that passes
through it.

This line will be the trace of a plane containing the
given line, and the point attained will be a point of
distance of the given line.

If now a second line be drawn parallel to this trace,
through the initial-point of the given line, it will be the
initial-line of this plane, and distances laid off upon it
from the initial-point may be transferred to the perspec-
tive of the given line by drawing lines to the perspec-
tive of the point of distance.

421. When the given line is inclined to the Axis, the distance from the station-point to the given vanishing-point is the hypotenuse of a right-angled triangle, of which the station-point is the vertex, the Axis the height, and the distance from the Centre to the given vanishing-point the base. See Fig. 134.

The distance of the van-ishing-point from the sta-tion-point. Fig. 134.

When the given line is normal to the plane of the picture, having the perspective of its vanishing-point at the Centre, the distance of this vanishing-point from the station-point is equal to the length of the Axis, and both isosceles triangles are right-angled. See Fig. 135.

Fig. 135.

422. Since the auxiliary plane may be taken in any direction, its trace may be drawn in any direction through the perspective of the vanishing-point. Any existing line that happens to pass through the perspective of the vanishing-point is accordingly available as such a trace, a line parallel to it being drawn through the initial-point as an initial-line. *Conversely*, any existing line that happens to pass through the initial-point may be used as an initial-line, and a trace may be drawn through the vanishing-point parallel to it, as has already been said (413).

Existing lines avail-able as traces or as initial-lines.

423. The *locus* of the perspective of the point of distance is then a circle described about the perspective of the vanishing-point as a centre, with a radius equal to the distance of this centre from the station-point. Such a circle will be cut by any given trace in two points. Each trace then

The *locus* of the point of distance. Fig. 136.

contains two such points of distance, one of which makes, with the station-point and the perspective of the vanishing-point, an obtuse-angled isosceles triangle, and the other an acute-angled one. This last is the one commonly employed. See Fig. 136.

When the given line is normal to the picture, the *locus* of the point of distance is a circle described about the Centre, with a radius equal to the Axis. The two points of distance make, with the station-point and the perspective of the point of distance, similar and equal isosceles triangles, right-angled at the Centre, one of which is, in general, as convenient to make use of as the other. See Fig. 135.

CHAPTER XVIII.

GEOMETRICAL PROBLEMS.

424. LET us now apply the propositions contained in the preceding chapter to the solution of the geometrical problems involved in the chapters that have gone before, adding some problems of a more purely speculative interest, which, in the ordinary practice of perspective drawing, seldom present themselves. We shall find that most of the common problems of Descriptive Geometry, those at least which concern the relations of right lines and planes, are easily solved in Perspective.

425. The following table presents the substance of the principles rehearsed in the previous chapter in a condensed form, as maxims : —

Maxims.

a. A plane passed through the station-point parallel to a plane cuts the plane of the picture in the trace of the plane (399). Fig. 124.

b. A line passed through the station-point parallel to a line pierces the plane of the picture at the vanishing-point of the line (407). Fig. 126.

Hence : —

c. Lines and planes passed through the station-point and through the vanishing-points and traces of given lines and planes make the same angles with each other as do the given lines and planes (400, 408). Figs. 120 and 127.

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d. If a line is normal to a plane its vanishing-point lies in a line
drawn through the Centre at right angles to the trace of the
plane.

[The theory of projections teaches that if a line is normal to a
plane its projection upon a second plane is at right angles to
the line in which the two planes intersect.]

e. The initial-line of a plane is parallel to its trace (398).

f. Parallel planes have the same trace, and their initial-lines are
parallel to it and to each other (398).   Fig. 119.

g. Parallel lines have the same vanishing-point (406).   Fig. 126.

h. Lines lying in or parallel to a plane have their vanishing-
points in its trace (412).   Figs. 115 and 129.

i. Lines lying in a plane have their initial-points in its initial-
line (412).   Figs. 115 and 129.
Conversely : —

j. The trace of a plane passes through the vanishing-points of all
the lines (*i. e.*, of any two lines) that lie in or are parallel
to it (412).

k. The initial-line of a plane passes through the initial-points of
all the lines (*i. e.*, of any two lines) that lie in it (412).

l. A line parallel to the plane of the picture has its perspective
parallel to itself and to the initial-lines and traces of all planes
that pass through or are parallel to it (409).   Fig. 121.
Conversely : —

m. All the planes that pass through or are parallel to a line that
is parallel to the plane of the picture have their traces and
initial-lines parallel to it and to its perspective (414).

n. The line of intersection of two planes has its initial-point and
vanishing-point at the intersection of their initial-lines and
traces (412).   Fig. 130.
Conversely : —

o. The initial-lines of all the planes that pass through a line
intersect each other at its initial-point (412).

p. The traces of all the planes that pass through or are parallel to a line intersect each other at its vanishing-point (412).

q. The Centre is the projection of the station-point upon the plane of the picture and the vanishing-point of lines normal to it (393, 411). Fig. 128.

r. The length of the Axis is the distance of the station-point from the centre (393). Fig. 128.

Notation.

426. The following system of Notation will be adopted in this chapter. It is conformable to that used in the previous chapters, except that the Centre is designated by V^C instead of by C, C being used to denote the direction of lines that are normal to the plane of the picture, having their vanishing-point at the Centre, V^C : —

The plane of the picture	pp.
The station-point	S.
The station-point when revolved into the plane of the picture	S', S'', etc.

A line is designated by a Roman capital : —

Horizontal lines, right and left	R. R', etc. ; L, L', ,,
Vertical lines (to the zenith)	Z, Z', ,,
Oblique lines to the right, up and down	M, M', ,,
Oblique lines to the left, up and down	N, N', ,,
Lines at 45° (bisecting ■ right angle)	X, X', ,,
Lines normal to a given plane (axes of the plane)	T, T', ,,
Lines normal to the plane of the picture	C, C', ,,
Sunlight and shadow lines	S
Other lines	E, I, O, U, Y, etc

Finite lines, by small capitals	H, Z, C, R, L, ,,
The perspective of a point is designated by an italic letter	a, b, c, m, x, ,,
The vanishing-point of a line by V, with the letter designating the line written above it	V^R, V^L, V^M, ,,
Other vanishing-points	V, V', V'', ,,

The initial-point of a line by I, with the letter designating the line written

    above it   .   .   .   . I^R, I^L, I^M, etc

Other initial-points .   .   .   . I, I′, I″, ,,

The points of distance of a line by D, with the letter designating the line written

    above it   .   .   .   . D^R, D^L, D^M, ,,

Other points of distance .   .   .   . D, D′, D″, ,,

A plane is designated by two Roman capitals, designating two of its ele-

    ments .   .   .   . RM, LM, ,,

Horizontal planes   .   .   .   . RL, R′L′, ,,

Vertical planes   .   .   .   . RZ, LZ, ,,

Planes of the shadow of lines .   .   .   . SR, SL, SZ, ,,

The trace of a plane is designated by the letter T, followed by the letters desig-

    nating the plane   .   .   .   . TRN, TLM, ,,

The Horizon   .   .   .   . TRL, or HH.

The traces of inclined planes .   .   .   . H′H′, H″H″, etc.

Traces of shadows of lines .   .   .   . TSR, TSL, ,,

The initial-line of a plane is designated by the letter I, followed by the letters

    designating the plane.   .   .   .   . IRN, ILM, ,,

The Ground-line .   .   .   . IRL, or gl.

In the plates a point is shown thus   .   .   .   . ×

,, ,, ,,   a line   ,,   ,,   ,, ...   _____

,, ,, ,,   the initial-line of a plane is shown thus . . __ __ __ __ __

,, ,, ,,   the trace of a plane   ,,   ,,   ,,   . . . __ . __ . __ . __

.. ,, ,,   a constructive line   ,,   ,,   ,, ...

Data.

427.   The elements that enter into a perspective prob-
lem are, in space, the relative positions of the spectator,
the perspective-plane, and the plane of the picture, as
given by the position of the Centre and of the Horizon,
and the length of the axis of the planes; the size, posi-
tion, and attitude of the object, as given by the position,
magnitude, and directions of the lines and planes to be
represented; and, in the plane of the picture, the per-
spectives of all points and their projection upon that

plane, the perspectives of all finite lines, and the initial
and vanishing-points of the infinite lines in which they
lie, and the initial-lines and traces of all planes.

428. When, in any given case, all these things are
known, the discussion is complete. The solution of a
problem in Perspective consists in showing how, where
only some of these elements are given or assumed,
others can be determined.

429. Unless the contrary is expressly mentioned, it
will be understood that the perspective-plane is ver-
tical, and that the position of the Centre, V^c, that of
the Horizon, HH, and that of the Ground-line, $g\,l$, are
given, as is also the length of the Axis, SV^c, which
determines the position of the station-point in front of
the picture.

430. The scale is generally supposed to be known ;
that is to say, the dimension that shall be given, in the
drawing, to lines lying in the plane of projection. This
depends upon the relative distance of the plane of pro-
jection and the plane of the picture from the station-
point (94), or upon the scale adopted for the miniature,
or model, which is supposed to be substituted for the
object itself (396). In that case dimensions can be
given by figures. But in purely geometrical discussions
they are best given by lines.

431. A single point may be given by its co-ordinates,
by its perspective in a given line or plane, or by its per-
spective and its projection upon a given plane. These
all resolve themselves into the case of a point lying in a

given line. If the point is given by its rectangular co-ordinates, or lies in a plane normal to the plane of the picture, it lies in a line normal to the picture; if it lies in an oblique plane, it lies in an oblique line; if it is given by its projection on a given plane, it lies in a line normal to the given plane.

432. Distances are measured from the three planes passing through the Centre, as an origin of rectangular co-ordinates, the three axes of co-ordinates being the Axis, SV^c, normal to the plane of the picture, and passing through the Centre and station-point; the Horizon, HH, normal to the vertical plane which passes through the Centre and the station-point and which is normal to the plane of the picture; and the zenith and nadir line, ZZ, normal to the horizontal plane, and, like the Horizon, lying in the plane of the picture. These three planes, namely, the Horizontal plane, the Normal plane, and the plane of the picture, are called the *principal* planes.

I. *Problems of Direction.*

These problems are solved by the use of orthographic projections upon the plane of the picture and the principal horizontal plane, according to the common rules of projection. Most of the constructions take place in front of the plane of the picture, which serves as the vertical plane of projection. See Plate XXVI.

In this plate the Horizon, HH, the Zenith line, ZZ, and the Centre, V^c, are omitted when not needed for the constructions shown.

PROBLEM I. To find the vanishing-point of lines making a given angle with the horizontal-plane, their projection upon that plane making a given angle with the normal plane or with the plane of the picture.

Let the lines M of a given system make the angle α with the horizontal plane, and let their projections upon this plane make the angle β with the normal plane. Then a line, M, belonging to this system, Given α and drawn through the station-point, S, in front of β to find V^M. the picture, and having these relations to the principal planes, will pierce the plane of the picture in the common vanishing-point of the lines of the system (**b**).

∴ Let R be the horizontal projection of the line M, and let it be drawn horizontally from the station-point, S, at the angle β with the Axis, until it touches the Horizon at the point V^R. This is the vanishing-point of the horizontal projections of the lines M of the given system; that is to say, of the lines parallel to R. The required vanishing-point, V^M, will lie vertically above V^R in the plane of the picture, and the triangle, V^{M}V^{R}S, right-angled at V^R, and having R for its base and M for its hypotenuse, will have the angle at S, between M and R, equal to α (**c**). If now this triangle is revolved about its vertical side, V^{R}V^M, until it coincides with the plane of the picture, S will fall at D. If the line M is drawn in its revolved position, making at D the angle α with the Horizon, the point V^M is easily found at the intersection of M, in its revolved position, with the line erected vertically above V^R.

PROBLEM II. If the direction of the given system is

determined by the angles which its projection upon the normal plane makes with the Axis, instead of by the angle the line itself makes with the horizontal plane, the vanishing-point can be found by a process analogous to the preceding, as in the plate.

PROBLEM III. Given the direction of a plane, or sys-tem of planes, by the direction of their axes, — To find the trace of the system; *that is to say* : —

Given a line by its vanishing-point, V^T, —

To find the trace of planes normal to it.

∴ Find V^T, the vanishing-point of the axes of the planes (Prob. I. or II.).

The required trace will be at right angles to a line, $V^T V^C a$, drawn through this vanishing-point and the Centre, V^C; for so much of this line as lies between V^T and V^C is the projection upon the plane of the picture of a line drawn from the station-point to the given van-ishing-point, V^T, and the required trace is the line in which the plane of the picture is intersected by a plane drawn through the station-point normal to this line. The line of projection and the line of intersection are then necessarily at right angles to one another (**d**).

It remains only to find the point a.

So much of this line as lies between V^C and a is the projection upon the plane of the picture of a line lying in the intersecting plane at right angles with the first line. This line is drawn through the station-point par-allel to the given plane, and pierces the plane of the picture at the required point, a, in the required trace.

These two lines, lying in the air in front of the picture, form at the station-point the vertex of a right-angled triangle, right-angled at S, the hypotenuse of which is the line of projection, $V^T V^C a$.

If this triangle is supposed to be revolved into the plane of the picture around this hypotenuse, the Axis of the picture, $V^C S$, will fall at right angles to it at $V^C S'$. The point a, at the other end of the hypotenuse, is then easily found by setting off at S' the right angle, $V^T S' a$. The required trace may then be drawn through a, at right angles to $V^T V^C a$.

PROBLEM IV. *Conversely:* Given the trace of a plane, or system of planes, —

To find V^T, the vanishing-point of lines normal to the system, — that is to say, of their axes.

Given a plane to find its axis.

∴ This simply reverses the operations of Problem III.

If the axis is parallel to the plane of the picture, the trace of the system of planes lies at right angles to it, and passes through the Centre, V^C. If the system of planes is parallel to the plane of the picture their axes have the Centre, V^C, for a vanishing-point.

PROBLEM V. Given the trace of a plane and the vanishing-point, V^O, of a line lying in it, —

To find V^M, the vanishing-point of a second line, lying in the same plane, making a given angle, a, with the first, or perpendicular to it.

Since lines drawn from the two vanishing-points to the station-point must meet at the station-point in the given angle ($\mathbf{m}$), and lie in a

Given V^O and a to find V^M.

plane which cuts the plane of the picture in the given trace ($\mathbf{j}$), —

∴ Revolve this plane into the plane of the picture, about the given trace. The projection of the station-point will move from V^C in a line at right angles to the given trace, and the station-point will fall in its revolved position at S'', the distance, $S'' a$, being equal to $S' a$, determined as in the previous figure, $V^C S'$ being the length of the Axis.

If V^O is the given vanishing-point, $V^O S''$ will be the revolved position of the line drawn from this vanishing-point to the station-point and a line making with it the given angle α, will cut the given trace at the required vanishing-point, V^M, as in the figure.

If the angle α is 90°, the two lines will be at right angles.

PROBLEM VI. *Conversely:* Given the direction of two lines, O and M, by their vanishing-points, V^O and V^M, to find the angle they make with one another in space.

Given V^M and V^O to find a.

∴ This simply reverses the operations of Problem V.

PROBLEM V. A. If the given line, lying in the given plane, is parallel to the plane of the picture, so that its vanishing-point, V^O, is at an infinite distance, it will be parallel to the trace of the plane (**1**). The line drawn through the station-point parallel to the given line will accordingly also be parallel to the given trace, and may be drawn parallel to it through S'' in its revolved position, as in the figure. The angle α may then be set off as before.

The given line parallel to the picture.

PROBLEM VI. A. *Conversely:* By a reversal of this process we may ascertain the angle made by a given line, lying in a given plane, with another line lying in the same plane and parallel to the plane of the picture.

PROBLEM VII. Given the direction of two planes by their traces to find the angle between them. *To find the angle between two planes.*

[The angle made by two planes is the supplement of the angle made by their axes.]

∴ Find the vanishing-points of the axes of the two planes (Prob. IV.), and then find the angle made by these axes. (Prob. VI.)

PROBLEM VII. A. If the planes are normal to the plane of the picture, they make the same *When the planes are normal.* angle with each other as do their traces.

PROBLEM VII. B. If the elements in the two planes which are parallel to the plane of the picture *When their traces are parallel.* are parallel to one another, *i. e.*, if the traces of the planes are parallel, as in Fig. 123, Plate XXV., the problem may be solved by finding the angle between the planes themselves instead of that between their axes, as follows : —

Find the points *a* and *a'*, in the traces of the two planes, as in Prob. III. The angle *aSa'*, revolved into the plane of the picture at *aS'a'*, will obviously be the required angle.

PROBLEM VIII. Given the direction of a line by its vanishing-point, and that of a plane by its *To find the angle between a line and a plane* trace, to find the angle between them.

[This angle is the complement of that made by the line and the axis of the plane.]

∴ Find the vanishing-points of the axis of the given plane (Prob. III.), and then find the angle between this axis and the given line (Prob. VII.).

∴ Another way : Project the given line upon the given plane (Prob. XXIX.), and then find the angle between the line and its projection (Prob. VI.).

PROBLEM IX. Given the direction of a line, or system of lines, M, by the vanishing-point, V^M, —

To find (1) the distance from the station-point, S, ·to the vanishing-point, V^M, (2) a point of distance, D^M, and (3) the *locus* of the points of distance of the system.

(1.) The line from the station-point to the vanishing-

To find a point of distance and its *locus*.

point is the hypotenuse of a triangle, right-angled at the centre of the picture, V^C, of which the Axis, $V^C S$, is one side.   By revolving this into the plane of the picture around its base, $V^C V^M$, the distance, $S V^M$, is easily found.

(2.) This distance laid off, right or· left, upon the trace of any plane which contains M, and which consequently passes through V^M (**g**), gives the points, D^M, which are called the points of distance of M.   Their distance from V^M shows the distance of the station-point from V^M.

(3.) As the traces of all planes containing M pass through V^M (**j**), all possible points of distance of the system M will lie in a circle, of which V^M is the centre, and the distance, $V^M S$, the radius.   This circle is the *locus* of D^M.

∴ If the given lines belong to the system C, normal to the plane of the picture, their vanishing-point is the

Centre, V^C, and the radius of the circle, which is the *locus* of D^C, is the Axis, V^CS. The point of distance, D^C, accordingly, always shows at once how far the station-point is from the picture. The length of the Axis is equal to the distance, V^CD^C.

The operations by which the remaining problems are solved are conducted in the plane of the picture by means of vanishing-points, initial-points, traces, initial-lines, and points of distance which have been previously determined.

II. *Problems of Dimension and Position.*

PROBLEM X. To cut off given lengths from a line given in perspective by its initial-point, I^M, and its vanishing-point, V^M, —

∴ Draw through the initial-point a line, at random, to represent the initial-line of a plane passing through the line, and draw a parallel line through the vanishing-point to represent the trace of this plane (**a, j, e**). Find the *locus* of the points of distance of the given line (Prob. IX.). The points in which it cuts the assumed trace will be points of distance of the given line.

To measure off lengths on a perspective line.

Lay off upon the initial-line, right or left, at m and m', the lengths to be cut off upon the line given in perspective, and from the points thus ascertained draw lines *across* the given line to the opposite point of distance. The lengths cut off upon the given line will be the required perspective lengths.

For, in the figure, the line $V^M D^M$ is, by construction, parallel to $I^M m$, SV^M (in the air in front of the picture) to $I^M a$ (**b**), and SD^M (also in the air), to $m a$ (**b**). And since the triangle, $SD^M V^M$ (in front of the picture) is isosceles, $V^M D^M$ having been taken equal to SV^M, the triangle shown in perspective at $I^M m a$ is isosceles also, the corresponding sides of the two triangles being parallel, and $I^M a$ is equal to $I^M m$.

It is to be noticed that these triangles are reversed in position.

PROBLEM XI.   *Conversely:* Given a finite line in per-

To find the true length of a given perspective line.

spective, or two points, with the initial-point and vanishing-point of the infinite line in which they lie, —

To find the length of the line, or the distance apart of the two points.

Let a and a' be the given points, and I^M and V^M the given initial-point and vanishing-point.

$\therefore$ Through I^M and V^M draw parallel lines in any convenient direction.   These will represent the initial-line and trace of a plane containing the given points (**e, j, k**) (422).   Find a point of distance, D^M (Prob. IX.).

From D^M draw lines through a and a', until they intersect the initial-line at m and m'.

The distance $m\, m'$ is the true distance of the points $a\, a'$, or the length of the line $a\, a'$.

If in this and the preceding problem the given points or line lie in a line normal to the plane of the picture, then I^C, V^C, and D^C take the place of I^M, V^M, and D^M.

PROBLEM X. A. To cut off given lengths from lines parallel to the plane of the picture.

It is necessary, in order to fix the position of the given line, that the initial-point and vanishing-point of an auxiliary line intersecting the given line should also be given.

∴ Let p be a point in the given line, $p\,a$, and I and V the initial-point and vanishing-point of an auxiliary line intersecting it at this point. Draw parallel lines through I and V to represent the initial-line and trace of a plane containing both lines. They will be parallel to the given line (**1**), and any distances, Im, $m\,m'$, taken upon the initial-line, may be intercepted upon the given line at $p\,a$ and $a\,a'$ by lines parallel in space (**c**), drawn from I, m and m', to V, as in the figure.

PROBLEM XI. A. *Conversely:* Given a finite line parallel to the plane of the picture, with the initial-point and vanishing-point of an auxiliary line passing through it, —

To find the real length of the given line.

∴ This problem is simply the reverse of the preceding.

PROBLEM XII. To find the perspective of a point given by its co-ordinates.

∴ Let the origin of co-ordinates be the Centre, V^{c}, and let the horizontal, vertical, and normal co-ordinates of the point be H, Z, and C,

respectively.

The normal co-ordinate will lie in an infinitely long line, C, passing through the required point and piercing the plane of the picture at its initial-point, I^C. Hence : —

Lay off in the plane of the picture, horizontally and vertically, from the origin, V^C, the co-ordinates H and Z. They give the position of I^C, the initial-point of the normal line passing through the required point. Cut off upon this line the distance, C (Prob. X.). The point obtained will be the perspective of the required point, a, as in the figure at A. The point I^C is the orthographic projection of the point a upon the plane of the picture.

[The orthographic projection of a point upon the plane of the picture, the perspective of the point, and the centre of the picture, lie, of course, in the same straight line.]

The initial-line and trace of the auxiliary plane may be taken at random, as at A, or, as is generally more convenient, and as is shown at B, may be taken as coinciding in direction respectively with H and the Horizon, HH, or with Z, and the zenith line, ZZ, as shown in the figure at C.

PROBLEM XIII. *Conversely :* Given a point by its perspective, a, and its projection on the plane of the picture, I^C, to find its co-ordinates.

To find the co-ordinates of a given point.

The horizontal and vertical distance of I^C from V^C, in the plane of the picture, are the co-ordinates H and Z, and the real length of the line $I^C a$ (Problem XI.) is the third co-ordinate, C.

III. *Problems of Planes, Lines, and Points.*

PROBLEM XIV. Given a plane by its initial-line and trace, —

To draw a line in it, either at random, or having a given direction parallel or inclined to the plane of the picture, or parallel to a given line lying in or parallel to the given plane, or making a given angle with such a line, and to find the vanishing-point and the initial-point of the line so drawn. To draw a line in a plane.

In all these cases the vanishing-point, which necessarily lies in the trace of the given plane (**h**), either may be assumed, or is given, or may be found by Problem V.

The initial-point necessarily lies in the given initial-line (**k**).

If the line is required to pass through a given point, *a*, —

∴ Draw the perspective of the required line through the perspective of the given point and the vanishing-point. The initial-point is at the intersection of this perspective line with the given initial-line (**i**). Through a given point.

If the line is required to pass through two points, or a finite line given by its terminal points, —

∴ Pass the perspective of the line through the perspective of the given points; its initial-point and vanishing-point will lie at its intersection with the initial-line and trace of the given plane, respectively. Through two points.

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If the line is required to be parallel to the plane of
the picture, its perspective will lie parallel to the
trace of the given plane (1), and its initial-point and
vanishing-point will be at an infinite distance.

Parallel to
the picture.   PROBLEM XV. *Conversely :* Given a line by
two points, or by its initial-point and vanish-
ing-point, —

To pass a plane through it at random, or parallel to a
To pass a
plane
through a
line.   given line, or normal to a given plane, or to
the plane of the picture.

If the line is given by two points, they are
given, virtually, as lying in parallel normal lines
(Prob. XII.).   The line, then, lies in a normal plane,
and its initial-point and vanishing-point are easily
found (Prob. XIV.) (**h, i**).

If the required plane is to be normal to a plane it
will be parallel to the axis of the plane ; if normal to
the plane of the picture it will be parallel to the axis
of the picture.   These conditions are comprised, then,
in the condition that the required plane shall contain a
line given by its initial-point and its vanishing-point,
Parallel to ▪
given line.   and shall be parallel to a second line, the
vanishing-point of which is either given or
easily ascertained (Prob. IV.).   Hence : —

∴ Draw a line through the vanishing-point of the
given line and the vanishing-point of the line to which
the required plàne is to be parallel, for the trace of the
required plane, and draw a second line, parallel to this
trace, through the initial-point of the given line, for the
required initial-line.

If it is normal to the plane of the picture the trace will pass through the centre, V^c. *Normal to the picture.*

If the required plane is to be drawn through the given line at random, any two parallel lines drawn through its initial-point and vanishing-point will represent, respectively, the initial-line and trace of a plane passing through it, as in Problems X. and XI. (413). *At random.*

PROBLEM XVI. To pass a line through a point given by its perspective, and lying in a plane given by its initial-line and trace, that shall lie in the given plane and shall be parallel to a second plane. *To draw in a plane a line that is parallel to a second plane.*

The required line will lie in one plane and be parallel to the other. Accordingly its vanishing-point will lie in both their traces (**h**); that is to say, at the intersection of their traces. Hence:—

∴ Draw a line from the intersection of the traces of the two planes, as a vanishing-point, through the perspective of the given point to the initial-line of the plane in which it lies. The point in which it meets this line will be the initial-point of the required line.

PROBLEM XVI. A. If the second plane is parallel to the first, the line can of course be drawn in any direction.

PROBLEM XVI. B. If the second plane is parallel to the plane of the picture the required line will also be parallel to the plane of the picture, and will be drawn parallel to the trace of the plane in which it lies (**l**).

PROBLEM XVI. C. If the two planes, though not parallel, have those elements parallel which are parallel to the plane of the picture, so that the traces of the two planes are parallel, as in Problem VII. B, and in Fig. 123, Plate XXV., the required line will still be drawn parallel to the trace of the plane in which it lies. It will then still be directed towards the infinitely distant intersection of the two parallel traces (1).

PROBLEM XVII. Given a point by its co-ordinates, or by its projection, or by its perspective and a line or a plane in which it lies, —

To pass a line through a point in space.

To pass a line through it at random, or having a given direction parallel or inclined to the plane of the picture, or parallel to a given line, or normal to or making a given angle with a given line in a given plane, or normal to a given plane, or to the plane of the picture.

All of these different ways of determining a point in position have been reduced by the previous problems to the case of a point given by its perspective and by the initial-point and vanishing-point of a line passing through it. So, also, all the conditions attached to the required line have been reduced to the conditions that its vanishing-point is known; that is to say, it is either assumed, is given, or may be found.

The problem is then simply this : —

Given a point by its perspective, a, and the initial-point, I^M, and vanishing-point, V^M, of a line passing through it, —

To pass a line, O, through it in the direction given
by a second vanishing-point, V°, and to find
I°, the initial-point of the required line.

In a given
direction.

∴ Pass a plane through the given line parallel to the
required line (Prob. XV.). The given point will lie in
this plane.   ▪

In this plane draw a line through the given point
having the required direction (Prob. XIV.).

PROBLEM XVIII. To pass a line through two points
given by their co-ordinates, or otherwise, and to
find its initial-point and its vanishing-point.

To pass a
line through
two points.

Each point lies, or may be found to lie, in a given
line normal to the plane of the picture.   The two
given points lie, then, in a given normal plane, and the
initial-point and vanishing-point of a line joining them
will lie in the initial-line and trace of that plane
(Prob. XIV.).

PROBLEM XIX. Given three points by their co-ordi-
nates, or otherwise, —

To pass a plane through them :

To pass a
plane
through
three points.

∴ Pass a line through the first and sec-
ond, and another through the second and third (Prob.
XVIII.).

Pass the required plane through the first line parallel
to the second (Prob. XV.).

PROBLEM XX. Given two lines, in the same plane,
to pass a plane through them, and find their
point of intersection.

To pass a
plane
through two
lines.

∴ Draw a line through V and V′, the

vanishing-points of the two lines, for the trace of the
plane, and through their initial-points, I and I', for its
initial-line. The intersection of their perspectives will
be the perspective of their intersection.

[The condition that the lines lie in the same plane
necessarily implies that they intersect one another,
or will do so if prolonged, and that the line joining
their initial-points is parallel to the line joining their
vanishing-points.]

PROBLEM XXI. *Conversely :* to discover whether two

To determine
whether two
lines inter-
sect.

lines given in perspective intersect in space;
that is to say, whether they lie in the same
plane : —

∴ If the line joining their initial-points is parallel to
the line joining their vanishing-points, they do ; other-
wise, not.

PROBLEM XX. A. If the lines are parallel to one an-

To pass a
plane
through two
parallel lines.

other, the line joining their initial-points will
be the initial-line of the required plane, and
its trace will be drawn parallel to the initial-
line through their common vanishing-point. There will,
of course, be no point of intersection.

B. If one of the lines is parallel to the plane of the

To pass a
plane
through two
lines when
one is parallel
to the
picture.

picture it will be parallel to the initial-line
and to the trace of any plane that passes
through it (1). The required initial-line and
trace will accordingly be drawn parallel to this
line through the initial-point and vanishing-point of the
other line.

C. If both lines are parallel to the plane of the

picture, an auxiliary line must be drawn from a point in the first line to a point in the second, and *The same, when both lines are parallel to the picture.* its initial-point and vanishing-point found (Prob. XVIII.). The initial-line and trace of the required plane may then be drawn through the initial-point and vanishing-point of this auxiliary line, parallel to the given lines.

PROBLEM XXI. A. *Conversely:* to discover whether two lines given in perspective intersect with *To determine whether two lines intersect when one is parallel to the picture.* one another, one being an oblique line, given by its initial-point, I, and its vanishing-point, V, and the other a line parallel to the plane of the picture, given in position by the projection upon the plane of the picture at I^c, of some point in it, a, the centre, V^c, being also given.

∴ Draw through a, the perspective of the given point, its line of projection from V^c to I^c, and through the same point an auxiliary line, parallel to the given oblique line, from V to I', the initial-point of the auxiliary line. If the given lines intersect, that is to say, lie in the same plane, this new initial-point will lie in the initial-line of that plane, which will be parallel to the parallel line (**m**). I' will accordingly be as far from the parallel line as is I.

If now the line I'V really passes through the point a, it will meet at that point the line of projection, $I^c V^c$, and lie in the same plane with it, a plane normal to the plane of the picture. The trace of this plane will pass through V and V^c, and its initial-line through I' and I^c, and these two lines will be parallel (**e**), as at A.

But if the given lines do not meet in space, the aux-
iliary line, I'V, will not really pass through the point
whose perspective is given at a, but will pass in front
of it, or behind it, and the lines VV^C and $I'I^C$ will not
be parallel, as at B.

PROBLEM XXII. Given two planes, by their initial-
To find the   lines and traces, —
intersection
of two planes.   To find their line of intersection.

∴ Draw a line from the intersection of the two initial-
lines, as its initial-point, to the intersection of the two
traces, as its vanishing-point (**n**).

PROBLEM XXII. A. If the two planes, as in Problem
          VII., B, and in Problem XVI., C, have those
The same,
when their elements parallel which are parallel to the
traces are
parallel.   plane of the picture, so that their traces
and initial-lines are all parallel, like those shown in
Figs. 119 and 123, Plate XXV., their line of inter-
section will be parallel to their traces and· initial-
lines, being directed to their infinitely-distant points
of intersection.

To find the position of this line an auxiliary plane
must be passed across the two given planes, and its
lines of intersection found. The point in which these
lines of intersection meet will be one point in the in-
tersection of the given planes, and the required line of
intersection may be drawn through the point parallel to
the given traces and initial-lines.

Such an auxiliary plane will have its initial-line, of
course, parallel to its trace ; and as such a plane may

take any direction, any two parallel lines may be held to represent its initial-line and trace. Hence : —

To find the required line of intersection of two planes whose traces and initial-lines are all parallel, —

∴ Draw any line across the given traces as the trace of the auxiliary plane, and any other line parallel to the first across the given initial-lines as its initial-line. Draw the two lines in which the auxiliary plane intersects the given planes (Prob. XXII.), and through the point where they cross one another draw the required line parallel to the given traces and initial-lines.

PROBLEM XXIII. Given three planes, —

To find their point of intersection.

∴ Find the line of intersection of the first and second, and the line of intersection of the second and third (Prob. XXII.). Then find the point of intersection of these two lines of intersection (Prob. XX.).

To find the point of intersection of three planes.

PROBLEM XXIV. Given a line and a plane, —

To find the point in which the line pierces the plane.

∴ Pass through the given line, an auxiliary plane, at random (Prob. XV.), and find its line of intersection with the given plane (Prob. XXII.). This line of intersection, and the given line, will both lie in the auxiliary plane. Find their point of meeting (Prob. XX.). This point will lie in the given line, and also in the given plane, and will be the point of puncture required. The auxiliary plane has

To find where a line pierces a plane.

been taken as vertical, this being, in the majority of cases, the most convenient position in which to assume such a plane.

PROBLEM XXIV. A. If the given line is parallel to the plane of the picture, its position is fixed by that of some point in it, and a normal plane may be passed through it, as in Problem XV.

The same, when the line is parallel to the picture.

IV. *Problems of Projection, etc.*

PROBLEM XXV. Given a point by its perspective, a, and the vanishing-point and initial-point of the normal or oblique line in which it lies, —

To find the projection of a point upon a plane.

To find its orthographic projection upon a given plane.

∴ Find the vanishing-point of lines normal to the given plane (Prob. IV.), and draw through the given point such a line (Prob. XVII.).

Find the point in which this normal line pierces the given plane (Prob. XXIV.).

This point is the required projection. The normal line is the line of projection.

PROBLEM XXV. A. If the given plane is parallel to the perspective plane it will be given in position by the perspective of some point in it with its projection upon the plane of the picture. The centre, V^C, will be the vanishing-point of the line of projection (**q**).

The same, when the plane is parallel to the picture.

∴ Draw the line of projection of the point in the given plane, and also the line of projection of the point in the given line, and find their initial-points (Prob.

XVII.). The two lines of projection will lie in a normal plane, the initial-line of which will pass through their two initial-points. This normal plane will intersect the given plane in a line parallel to this initial-line, and the point in which this line cuts the line of projection of the point in the given line, is the point required.

PROBLEM XXVI. Given a line in perspective, to find its projection upon a given plane.

∴ Find the projection upon the given plane of some point of the line (XXV.).

To find the projection of a line upon a plane.

Pass through the given line an auxiliary plane normal to the given plane (Prob. XV.). Find the line in which this auxiliary plane intersects the given plane (Prob. XX.). This line is the required line of projection.

PROBLEM XXVII. Given a point and a line in perspective, —

To find upon the given line its point of nearest approach to the given point, and the distance between them.

To find the distance between a point and a line.

∴ Pass through the given point a plane normal to the given line (Prob. III.), and find the point in which the given line pierces this plane (Prob. XXIV.). This point is obviously the required point, and its distance from the given point (Prob. XI.), the required distance.

PROBLEM XXVIII. Given two lines in perspective, to find their points of nearest approach, and the distance and direction from one another of those points.

To find the distance between two lines.

∴ Pass through one line an auxiliary plane parallel
to the other (Prob. XV.), and find the projection of the
second line upon this plane, to which it is parallel
(Prob. XXVI.). This projection and the first line will
both lie in this plane.

Find the point in which the projection of the second
line intersects the first line (Prob. XX.). This is one of
the required points, and the point of the second line
thus projected is the other.

Find the length of the line joining these two points
(Prob. XI.) ; this is the required distance of the two
points, and the direction of the line joining them, which
is the direction of lines normal to the auxiliary plane, is
the required direction.

PROBLEM XXIX. Given the perspective of a finite
line, and its direction by the vanishing-point of the line
in which it lies : —

To divide a
line in any
given ratio. To divide it in any desired ratio, or into
equal parts, or proportionally to another line.

[This problem is analogous to the problem of cut-
ting a given length from a perspective line (Prob. X.).
But in that case the given line is indefinite in length,
and the auxiliary triangle is an isosceles triangle, of
which the adjacent sides are divided into equal parts by
lines drawn parallel to the base. In the present case
the perspective line is of definite length, and the aux-
iliary triangle is scalene, the adjacent sides being divided
proportionally.]

∴ Through the given vanishing-point, draw a line in

any convenient direction as the trace of a plane passing through the given line (**j**).

Through either end of the given finite line draw a line parallel to the trace thus assumed, and lay off upon this line, as a line of measures, the given proportional or equal parts at any convenient scale.

Draw through the extreme point thus ascertained a line passing through the other end of the given finite line, and prolong it until it intersects the assumed trace. Towards this point of intersection, as a vanishing-point, draw lines from the points taken upon the line of measures. The points at which they cut the given line will be the required points of division.

PROBLEM XXIX. A. If the given line is parallel to the plane of the picture, its perspective will be proportional in all its parts to the given line, and it may be divided, as the line itself would be, according to the methods of plane geometry.

The same, when the line is parallel to the picture.

CHAPTER XIX.

THE PRACTICAL PROBLEM.

433. AFTER all, the question remains, How is one to

The data.

go to work, in a given case, to make a per-
spective drawing. The shape and size of the
object to be drawn, a building for instance, are, of
course, supposed to be given, with the scale to be em-
ployed in the plane of the picture. The scale either
may be assumed or may be determined by comparing
the relative distances of the object and of the plane of
the picture from the spectator (94).

434. The next thing to be determined is the *attitude*

The attitude
of the object.

of the object; that is to say, the angle its
principal lines shall make with a line drawn
from the eye to the object. The direction of this line
is in general purely arbitrary, being so chosen as to
exhibit the building or other object in its best aspect.
The plane of the picture is generally taken at right
angles to this line, which then becomes the Axis of
the picture, some point near the middle of the object
being then at the Centre. But when it is possible, by
giving this Axis a slightly different direction, to bring

the principal lines of the object at 45° with the plane of the picture, making the Centre coincide with V^X, the "vanishing-point of 45°," it is best to do so. This adjustment is exemplified in Fig. 14, Plate V., and in Fig. 138, Plate XXVII.

Best at 45°,

435. When one side of the object is nearly parallel with the picture, it is often desirable, and sometimes necessary, as has been explained in paragraph 276, to make it exactly parallel.

or parallel.

436. It is generally desirable to have the plane of the picture as far from the eye as possible, which is equivalent, for any given scale, to having the object itself as distant as possible, the distance of the picture multiplied by the denominator of the fraction expressing the scale, giving the distance of the object (94). If, for instance, the scale employed in the drawing is that of one eighth of an inch to the foot, that is to say, one ninety-sixth full size, the object, or that portion of it that lies in the plane of projection, or plane of measures, will be ninety-six times as far away as the drawing.

The Axis to be as long as possible.

The further the station-point is from the picture the easier will it be for the spectator to occupy it, and the less will be the apparent distortion of the drawing if he fails to occupy it exactly.

437. But setting the station-point far from the picture is equivalent to setting the vanishing-points far apart, so that practically the first thing to do after the attitude of the object is

i. e., V^L and V^R to be as far apart as possible.

chosen, and the angle it is to make with the plane of the
picture determined, is to fix the distance apart of the
principal right-hand and left-hand vanishing-points, V^R
and V^L. These points, which of course lie in the Hori-
zon, are generally set at the extreme limits of the table
or drawing-board upon which the work is to be done.
See Fig. 137, A, Plate XXVII., in which the assumed
attitude of the object is shown by two lines drawn at
right angles to one another, making the given angles
with the picture. These lines may be drawn in any
convenient place, it makes no difference where.

438. As the principal horizontal lines, R and L, van-
Plate XXVII. ishing at V^R and V^L, upon the Horizon, are
Fig. 137. generally at right angles, the station-point S
is, in plan, generally at the vertex of a right-angled
triangle, of which the line $V^L V^R$, in the plane of the
picture, is the hypotenuse. The *locus* of S is accordingly
a horizontal semicircle, of which the line $V^L V^R$ is the
diameter. The next step after fixing these points is,
then, to describe such a semicircle, and to find upon it
S, V^C, V^X, the point S, such that the line SV^R will be
D^L, D^R, and
D^X. parallel to the right-hand side of the object,
and SV^L to its left-hand side. A perpendicular dropped
from this point upon the line $V^L V^R$ will give the posi-
tion of the Centre, C (or V^C); a diagonal line bisecting
the right angle will give the point V^X, the "vanishing-
point of 45°" (44); and the lines SV^R and SV^L revolved
into the plane of the picture will give respectively
the right and left-hand points of distance, D^R and D^L.

D^X, the point of distance of the diagonal line, may be obtained at the same time, if desired, by revolving SV^X into the plane of the picture, as in Plate IV., Fig. 11.

439. If shadows are to be cast, and the vanishing-point of shadows, V^S, lies beyond either of the principal vanishing-points, as it does in the figure, room must be allowed for this vanishing-point also. This space, however, may be saved by taking the sun in the plane of the picture, as in Fig. 36, Plate VIII., with the vanishing-point of shadows at an infinite distance (184).

440. If the object is set just at 45°, as has been recommended, and as is done in Plate V., and in Fig. 138, Plate XXVII., its two sides making equal angles with the plane of the picture, it is not necessary to describe the semicircle at all. The Centre will be half way between V^L and V^R, the station-point will be the same distance in front of the Centre, V^X will coincide with V^C (or C), and D^L and D^R may be found as before. They will be almost exactly two fifths of the distance from the Centre to either vanishing-point, as shown. *The object at 45°. Fig. 138.*

441. All these operations are conducted in plan, the paper at 137, A, and 138, A, representing the ground plane, or horizontal plane of projection, the line $V^L V^R$ being the projection of the Horizon. This line also represents both the plane of the picture, *p p*, seen edgewise, and the ground line, *g l*, in which the plane of the picture cuts the ground plane. *The ground plane.*

442. If now the plane of the picture is revolved into

20

the plane of the paper about the Horizon, the points

V^R, V^L, V^X, D^R, D^L, and C (or V^C) will retain
their positions, the ground line, $g\,l$, will appear
in the plane of the paper at some distance below the
Horizon, and parallel to it, and the station-point, S, will
be in the air in front of the picture, opposite the Centre,
as shown in Plate XXV., and in Fig. 13, Plate IV.

443.   Vertical lines, erected in the plane of the picture

at V^L and V^R, will now establish TRZ and TLZ,
the traces of the principal vertical planes, and
the vanishing-points of the inclined lines M, M′, N, and
N′, lying in or parallel to these planes, may be fixed
by drawing lines at D^L and D^R, that make, with the
Horizon, the same angles, α and β, that the lines them-
selves make with the horizontal plane.   The points in
which these lines intersect the traces of the vertical
planes will give the vanishing-points V^M, $V^{M\prime}$, V^N, and
$V^{N\prime}$ (88).

444.   These points being determined, the traces of

inclined planes, TRN, TRN′, TLM, and TLM′,
can be drawn whenever they are needed, and
V^P and $V^{P\prime}$ the "vanishing-points of hips and valleys,"
can be ascertained, as in Plates I., II., III., IV., and V.

445.   In the same way V^S, the vanishing-point of

sunlight or of shadows, may be fixed by de-
termining first a point, $V^{R\prime}$ (or $V^{L\prime}$), the van-
ishing-point of the horizontal projection of the rays of
light, as in Fig. 137, A. $D^{R\prime}$ (or $D^{L\prime}$) can then be obtained
by revolving the horizontal line $SV^{R\prime}$ (or $SV^{L\prime}$) into

the plane of the picture. The position of V^s, immediately above or below $V^{R\prime}$ (or $V^{L\prime}$), can then be fixed by laying off at $D^{R\prime}$ (or $D^{L\prime}$) the real angle made by rays of light with the ground-plane, and the traces of the planes of invisible shadows, and the vanishing-points of the lines of visible shadow obtained by drawing lines from V^s to the various vanishing-points, and noting their intersection with the traces previously determined, as is done in Figs. 34, 35, 36, and 38, Plate VIII. (181).

446. The station-point, S, and the principal vanishing-points, V^R and V^L, being once established, all these other traces, vanishing-points, and points *Preliminary operations.* of distance may be determined either at once, as in Fig. 38, before the drawing is begun, or from time to time, as they are needed, during the progress of the work. In either case they are the necessary scaffolding, so to speak, without which the various constructions cannot be carried on.

447. If the drawing is to be made in Parallel Perspective the extreme vanishing-points become *Parallel perspective.* the vanishing-points of diagonals, which will coincide with the points of distance, as in Plate VI. It will generally be convenient to establish at once points of half or quarter distance, as shown in Fig. 21, in that Plate (142), and in Fig. 94, Plate XX. (333).

448. All this preliminary work is concerned solely with the *direction* of lines and planes, not with *The position of the object, horizontally.* their position. Before constructing a perspec-

tive drawing by their aid it is necessary to determine
also the *position* to be assigned to the object; that is to
say, to some prominent point in it. The point generally
selected is the lower end of the nearest corner.

The first thing to be done is to determine how far to
the right or left of the Centre this point shall be set.
It is generally on the right if the left-hand side of the
object is to be made prominent, and *vice versa*. In Fig.
137 the position of the front corner is assumed, and, the
building being rather a large one, it is set considerably
to the right of the Centre. In Fig. 138 both the attitude
and the position of the building are determined upon
the orthographic plan at A.

449. As everything in a perspective drawing is more
or less distorted, according as it is more or less removed
from the Centre, opposite the eye (260), it is of the
first importance that the object shall be as near the
Centre as possible. The Centre does not necessarily, as
has been said, coincide with the middle of the picture.
But it is desirable to have it do so, as nearly as is con-
venient; and, as the object itself naturally occupies the
middle of the picture, if the Centre is in the middle of
the object this point also is generally attained.

450. But it is often worth while to throw the object
The Centre. considerably to one side of the Centre; that is
to say, to have the Centre quite out of the
middle of the picture, if by so doing the practical ad-
vantages and convenience of having the principal right
and left-hand vanishing-points at 45° can be secured.

In Fig. 15, for instance, Plate V., the Centre is near the left-hand corner of the little house. But the main lines of the building do not materially differ from those of Fig. 12, Plate IV., in which the Centre is near the right-hand corner.

451. Where, as in Fig. 15, two objects are to be shown, it is of course impossible that both should occupy the same place at the Centre. *Two objects.* But it is generally practicable to have one of them, or both of them, if, as in this case, they are parallel to one another, at 45° with the picture.

452. The position to be given to the object, horizontally, having been determined, the next thing is to draw a Perspective Plan of it; *i. e.*, to *The perspective plan.* put into perspective its horizontal projection.

453. The horizontal plane upon which the perspective plan of the object to be represented is sup- *The ground-* posed to be drawn, is called the Ground-Plane, *line, g l.* and its initial line, *i. e.*, the line in which the ground-plane cuts the plane of the picture, is called the ground-line, or line of horizontal measures, as has been said. It is convenient, for many reasons, to have this as far as may be below the Horizon (46, 101), and it is well to draw it upon a separate piece of paper, covering the lower part of that upon which the drawing is to be made, *The sunk-* *perspective* so that the construction lines that lie in its *plan.* neighborhood may not deface the picture, and so that they may be removed and used again, if necessary, instead of being erased.

454. This is shown in Figs. 137, B, and 138, B, in which the Horizon, with the various vanishing-points and points of distance, are transferred directly from Figs. 137, A, and 138, A, and the ground-line, *g l*, drawn in an inch or two lower down.

In practice the figures A and B would be drawn one over the other, on the same paper. But in the plate the constructions by which the position of the vanishing-points and points of distance is determined, both upon the horizontal plan and in the plane of the picture, are drawn at A, and the perspective itself, and the perspective of the plan, both of which are in the plane of the picture, are drawn at B. This avoids confusion, and allows all the construction lines to be shown in full, which in practice would be drawn only in part.

455. It is customary to have the front corner of the building, or other object to be drawn, lie in the plane of projection, or, which comes to the same thing, to The starting-point. have the imaginary model touch the plane of the picture (396) as in these figures. In the perspective plan, then, the horizontal projection of this corner will lie in the ground-line, as shown at the point I. Lines drawn from this point, as an initial-point, to the principal horizontal vanishing-points, V^R and V^L, are the front lines of a perspective plan. They are infinite lines, upon which the horizontal dimensions of the object can be cut off by means of the points of distance already established, the ground-line serving as a line of horizontal measures.

456. The length of the right-hand side of the building, or other object, with its subdivisions, being Horizontal lines inclined to the picture. then laid off upon the ground-line to the right of this point, and of the left-hand side towards the left, may be transferred to these infinite perspective lines by drawing lines *across* them to the right-hand and left-hand points of distance respectively.

457. In drawing the perspective plan constant use may be made of the diagonal line bisecting the right-angle, and of its vanishing-point, V^x. vx

V^x, for finding the plan of hips and valleys and other lines lying at 45° with the principal lines. Further illustrations of this may be found in Plates II. and III.

If the principal horizontal lines of the perspective plan lie at 45° with the ground-line, as in Fig. 138, one set of the right-angles in which they meet will be bisected by lines drawn to the Centre, and the others by lines drawn parallel to the Horizon. The hips, also, on the right and left of the roofs, will be parallel to the picture, and will be drawn parallel to the traces of the planes in which they lie, as shown in the figure.

458. Dimensions taken by scale upon the ground-line may be transferred to lines lying in the Lines parallel to g l. horizontal plane and parallel to the plane of the picture, and accordingly parallel to the ground-line, by drawing lines to any point on the Horizon as a vanishing-point of parallel lines. The fence in Fig. 137, B, is drawn in this way.

459. As many different perspective plans may be
made as the complexity of the subject may
seem to require, and they may be above or
below the picture, as may be most convenient, as in
Plates III. and IV. This is illustrated also in Plate
XXVII., where Fig. 140 shows three perspective plans,
and Fig. 137 two. The work upon them may be done
all at once, or from time to time during the progress
of the drawing, as may be preferred. All the details
of the plans of every part may thus be put into per-
spective. But it is not of course necessary to com-
plete the plan of any parts that cannot be seen. In
general it suffices to make the plan of the two sides
that show, and of such more remote portions as are
visible above these sides.

460. The perspective plan being made, or at any rate
fairly begun, the drawing itself may be com-
menced. The perspective of the object itself
lies directly above the plan, but how far above depends
upon the relative altitude of the object and of the
spectator. The points on a level with the eye
will always, of course, be seen on the Ho-
rizon. So much of the object as is above the eye, per-
haps the whole of it, will appear above the Horizon;
whatever is below the eye will be drawn below the
Horizon. The *starting-point*, that is to say,
the lower end of the front corner, will lie
directly above the corresponding point in the perspec-
tive plan, and as far below the Horizon, by scale, as the

spectator's eye is supposed to be above the point itself, as at *c*, Fig. 137. The same perspective plan will serve to make several views of the same object, taken at different levels, above or below the starting-point, as is done in Figs. 4, 5, and 6, Plate II.

461. The perspective plan, drawn in the plane of the picture, suffices to determine all horizontal dimensions; that is to say, the position of all vertical lines.

The position of horizontal lines is determined by laying them off by scale upon the initial-line *The line of vertical measures.* of some vertical plane, as a line of vertical measures. When the nearest corner touches the plane of the picture it is generally used for this purpose. This line is then at once the initial-line of the right-hand vertical plane, R Z, and of the left-hand vertical plane, L Z, and serves as a line of vertical measures for both, as at *v v* in Figs. 137, B, and 138, B. The scale employed is the same as that used upon the ground-line for determining the horizontal dimensions of the perspective plan, since all lines in the plane of projection are drawn to the same scale (94).

462. But any plane occurring in the object may be prolonged until it cuts the perspective plane, *Several such lines* and have a line of measures of its own, in its own initial-line, as at *v' v'* in Fig. 137, B, which serves as an independent line of measures for the end of the wing. The vertical dimensions taken upon *Vertical dimensions* these lines of measures may be transferred directly to any vertical line which lies in this vertical plane, and which is accordingly parallel to the line of

vertical measures, by means of the vanishing-points V^R and V^L. In this way is determined the position of all the horizontal lines in Figs. 137, B, and 138, B, the vertical lines erected from the corresponding lines in the perspective plans serving to determine their length.

463. Moreover, just as such dimensions were transferred to lines lying in the ground-plane, but inclined to the picture, by means of a point of distance upon the Horizon, so dimensions taken by scale upon a vertical line of measures may be transferred to lines that lie in
Points of distance upon vertical traces. the vertical plane and are inclined to the plane of the picture, by means of points of distance taken upon the trace of the vertical plane in which they lie (113). This is illustrated in Fig. 12, Plate IV., where the heights of the gable are set off upon the vertical line through the corner of the house.

464. If any part of the object advances in front of the
Constructions in front of the perspective plane. Fig. 139. principal vertical planes, or, in plan, in front of the principal lines of the perspective plan, as is the case with the wing of the building shown in Fig. 137, its plan can be drawn in perspective by prolonging the leading perspective lines *in front* of the perspective plane, as is shown in Fig. 139. In this figure the dimensions to be set off upon this part of a left-hand line, L (or of a right-hand line, R), are set off upon the ground-line to the right of the initial-point instead of to the left (or to the left instead of to the right); and in transferring them to the perspective line they are brought forward away from the

point of distance, instead of being carried backward toward it, as before.

The length of the wing of the building in Fig. 137 is ascertained in this way: The dimension R^2, taken from the elevation above, is laid off upon the ground-line *to the left* of the point I, the initial-point of the perspective line, R, and is transferred to the prolongation of that line in front of the plane of projection and below the ground-line by means of the point of distance, D^R, as in Fig. 139. See also Fig. 15, Plate V. (109).

Another way of drawing such objects, or parts of an object, is shown in the second perspective plan at the bottom of the same figure. The point a, where the wing joins the main building, having been ascertained as before, by measuring off upon the principal left-hand line the distance, L^1, a right-hand line, directed towards the right-hand vanishing-point, V^R, is drawn through the point a until it intersects the ground-line at b, its initial-point. If now a second line be drawn through a, directed upon D^R, the right-hand point of distance, and cutting the ground-line at d, the distance, bd, intercepted upon the ground-line, will be the real length of the line ab, and the real length of the wing, R^2, may be laid off upon the ground-line from d and transferred to the line ab by means of D^R, as shown.

If parts of the object to be drawn are advanced not only in front of the principal planes, but in front of the plane of projection, as often happens with the cornices of buildings, and with steps and platforms, as is shown in Fig. 138, they may be put into the perspective plan

by the methods just described.   In this case the points
and lines in which the several lines and planes cut the
plane of projection are their initial-points and lines, and
may be set off by scale.   The points $a\,a$, at which the
eaves of the building in the figure, for example, pierce
the plane of projection, are equally far above the Ho-
rizon, and on a level with the top of the corner between
them.

465. Fig. 140, which is a view of the spire of the
church of St. Stephen's, Walbrook, illustrates
the use of several perspective plans, and also
the advantage of taking a plane of projection
considerably in front of the object instead of in contact
with it.   The extension of the right-hand and left-hand
vertical planes of the tower until they cut the plane of pro-
jection gives five initial-lines, or lines of vertical meas-
ures, on each side, all of which are quite outside of the
picture, instead of one in the middle of it, as is the case
when the front corner is taken as the line of measures.
These are lettered I^R, $I^{R\prime}$, etc., I^L, $I^{L\prime}$, etc. respectively.
This entirely frees the picture from constructive lines.

Setting the object some distance behind the perspec-
tive plane of course makes its perspective smaller, but
this may be met by setting off the dimensions upon the
lines of measures at a larger scale, which, when the
position of the vanishing-points remains unchanged, is
equivalent to moving the plane of the picture nearer to
the object itself.   In Fig. 140, the scale employed at B for
horizontal distances upon the ground-line, and for vertical

Fig. 140.
Two lines of
vertical
measures.

dimensions in the lines I^R, I^L, etc., is double that of the elevation at A, from which the dimensions are taken.

466. The same result may be produced by employing scales of vertical measures beyond the object, Small-scale in accordance with the theory of small-scale data. data discussed in a previous chapter (339). If these are set up half-way between the initial-lines, or lines of vertical measures, and their corresponding vanishing-points, the scale to be used will be half as large, as in the figure at $v\,v$, $v'\,v'$, etc., where the heights set off are the same as in the elevation alongside.

If both scales of height are used, as in the figure, one on the right of the picture and another, at half The vanishing-point the scale, on the left, the use of the left-hand unnecessary. vanishing-point, V^L, may be dispensed with, the perspectives of the horizontal lines being put in by drawing lines between the corresponding points on the two scales.

467. In the largest of the perspective plans employed in the figure, below the picture, advantage is taken of the fact that the two sides of the tower are exactly alike, to dispense also with the use of the point D^R. The points ascertained upon the left-hand side by means of the left-hand point of distance, D^L, are transferred to the right-hand side by means The method of the diagonal line directed towards V^X, the of diagonals. " vanishing-point of 45°," in accordance with the principle illustrated in Fig. 6, Plate II. (61).

This plan illustrates also the principle of Auxiliary Horizons discussed in section 365. But in- Auxiliary Horizons. stead of sinking the perspective plan in order

to prevent the angles of intersection from being too
acute, and accordingly putting the ground-line four or
five inches lower down, an auxiliary Horizon, H′ H′, is
drawn in four or five inches above the real Horizon, the
ground-line being retained, and the lines of the per-
spective plan are directed to the vanishing-points and
points of distance found upon this new Horizon, as in
Plates XXII. and XXIII.

468. Lines lying in an oblique plane can be measured
Oblique   off by means of a scale taken upon a line of
planes.   measures which is the initial-line of the plane
in which they lie, just as well as lines lying in hori-
zontal or in vertical planes.   This line will of course be
parallel to the trace of the plane drawn through the
vanishing-points of two of its elements (398).   If the
lines are parallel to the picture they will be parallel
both to the trace of the plane and to this line of meas-
ures, and dimensions taken upon the line of measures
may be transferred to the perspective line by drawing
parallel lines to any point upon the trace, as a vanish-
ing-point.   Dimensions taken upon the line of measures
may be transferred to lines lying in the oblique plane
and inclined to the picture, as before, by means of a
point of distance taken upon the trace of the oblique
plane.   See Fig. 15, Plate V. (111).

469. In the case of lines that lie at the intersection
Choice of   of two planes, it is a mere matter of con-
lines of
measures.   venience whether one shall treat them as
lying in one plane or in the other.   Either plane will

do, its initial-line serving as a line of measures. Its trace will contain the vanishing-point of the line in question, and its points of distance (412). All the points of distance of a line will be at the same distance from its vanishing-point, since the *locus* of its points of distance is a circle, of which the vanishing-point is the centre (423). See Plate V. (117).

470. These processes suffice not only to give the perspective of every point and line the position Choice of and direction of which is known, but to fur- methods. nish several methods by which they can be determined. Whether one or another of the methods shall be employed in a given case is a matter of discretion, in which the judgment and experience of the draughtsman must guide his choice. A line, for instance, may be determined in direction either by fixing the position of the points in it, or by finding its vanishing-point. It is sometimes more convenient to do one, sometimes the other. Whichever is employed, the other may be used to test the accuracy of the result.

471. In completing a perspective drawing, many special devices may be employed to abbreviate Special labor. Of these the most important are the devices. different ways of dividing lines in a given ratio, the different ways of casting shadows by natural or by artificial light, the use of points of half-distance or quarter-distance, and the various other devices for bringing the work within small limits, Mr. Adhémar's scheme for avoiding the difficulties experienced in drawing distant

objects, by the use of inclined perspective plans, with the suggested modifications, the employment of lines already existing as traces of auxiliary planes, and the special processes to be followed in putting circular arcs into perspective, with the practical adjustments to be made in the results. It is not necessary again to go into these details of procedure.

472. It is, however, worth while to say that there are
Mechanical aids. some mechanical devices not mentioned in the previous pages which are of service when vanishing-points are inconveniently far off. One of these is the employment of wooden or brass arcs, fastened to the drawing-board, upon which a T-square armed with pegs moves so as to direct its upper edge always to the centre of the arc, as a vanishing-point. Another way to effect the same thing is to cut a curve upon the handle of the T-square, and move it upon pins driven into the drawing-board. See Fig. 141.

473. A third device is to cover the paper with a net-work of lines, drawn in pencil, and directed towards the principal vanishing-points, as a guide in sketching. A useful variation of this is a paper carefully ruled in ink, to be used in sketching upon paper laid over the lines, and thin enough for them to be seen through it.

474. It may be added that the simplest way, in prac-tice, to obtain the position of the station-point in plan, and the direction of the principal right-hand and left-hand lines, R and L, when the principal vanishing-points, V^L and V^R, have been assumed (439), is to drive pins into the drawing-board at the vanishing-points, and then

. to hold two T-squares at right angles to one another, and move them upon these pins. The right angle in which they meet will of course sweep the board in a semi-circle, and if arrested at the point where the two edges of the T-squares have the required direction, will give the station-point.

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